International Journal of
***O**perations **R**esearch*

International Journal of Operations Research Vol. 8, No. 2, 27–32 (2011)

American Option Pricing Formula for Uncertain Financial Market

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Received September 2010; *Revised October 2010*; *Accepted January 2011*

Abstract—Option pricing is the core content of modern finance. American option is widely accepted by investors for its flexibility of exercising time. Uncertain finance is an application of uncertainty theory in the field of finance. In this paper, an American option pricing formula is derived for uncertain financial market and some mathematical properties are discussed. In addition, some numerical examples are proposed.
Keywords—finance, uncertain process, option pricing.

1. INTRODUCTION

In our daily life, we usually use language like "about 100km", "approximately 39^oC", "big size" to express some imprecise information and knowledge. Perhaps some people think that these imprecise quantities are subjective probability or they are fuzziness. However, a lot of surveys showed that they are neither like randomness nor like fuzziness. In order to develop a more general measure to model these imprecise quantities, Liu (2007) founded an uncertainty theory that is a branch of mathematics based on normality, monotonicity, self-duality, countable subadditivity and product measure axiom.

As an application of uncertainty theory, Liu (2009c) proposed a spectrum of uncertain programming which is a type of mathematical programming involving uncertain variables. Besides, Li and Liu (2009a) proposed uncertain logic, which can be seen as a generalization of multi-valued logics. Liu (2008) introduced an uncertain process as a sequence of uncertain variables indexed by time or space. As a counterpart of Brownian motion, Liu (2009a) designed a canonical process that is a Lipschitz continuous uncertain process with stationary and independent increments. Following that, uncertain calculus was initiated by Liu (2009a) to deal with differentiation and integration of functions of uncertain processes. In addition, Liu (2008) gave the definition of uncertain differential equation. After that, Chen and Liu (2010) proved an existence and uniqueness for uncertain differential equation. In order to provide a methodology for collecting and interpreting expert's experimental data by uncertainty theory, uncertain statistics was started by Liu (2010a) in 2010 in which a questionnaire survey for collecting expert's experimental data was designed and a principle of least squares for estimating uncertainty distributions was suggested. For exploring the recent developments of uncertainty theory, the readers may consult Liu (2010a).

In the early 1970s, Black and Scholes (1973) and, independently, Merton (1973) constructed a theory for determining the stock option price which is the famous Black-Scholes formula. Stochastic financial mathematics was founded based on the assumption that stock price follows geometric Brownian motion. As a different doctrine, based on the assumption that stock price follows a geometric canonical process, uncertainty theory was first introduced into finance by Liu (2009a) in 2009. Furthermore, Liu (2008) derived an uncertain stock model and a European option price formula. Besides, uncertainty theory was introduced to insurance models by Liu (2010b) based on the assumption that the claim process is an uncertain renewal reward process.

Option pricing is the core content of modern finance. American option is widely accepted by investors for its flexibility of exercising time. In this paper, an American option pricing formula is derived for uncertain financial market and some mathematical properties of them are discussed. The rest of the paper is organized as follows. Some preliminary concepts of uncertain process are recalled in Section 2. American option pricing formulae are derived and some properties of them are studied in Sections 3 and 4, respectively. Finally, a brief summary is given in Section 5.

2. PRELIMINARY

An uncertain process is essentially a sequence of uncertain variables indexed by time or space. The study of uncertain

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process was started by Liu (2008).

Definition 1. (Liu (2008)) Let T be an index set and let (Γ, L, M) be an uncertainty space. An uncertain process is a measurable function from $T\times(\Gamma, L, M)$ to the set of real numbers, i.e., for each $t\in T$ and any Borel set B of real numbers, the set

$$\{X_t \in B\} = \{g \in \Gamma \mid X_t(g) \in B\}$$

is an event.

An uncertain process X_t is said to have independent increments if

$$X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \mathbf{K}, X_{t_k} - X_{t_{k-1}}$$

are independent uncertain variables where t_0 is the initial time and $t_1, t_2, \mathbf{K}, t_k$ are any times with $t_0 < t_1 < \mathbf{K} < t_k$.

Theorem 1. *(Extreme Value Theorem, Liu (2010b)) Let X_t be an independent increment process and have a continuous uncertainty distribution $\Phi_t(x)$ at each time t. Then the supremum*

$$\sup_{0\le t\le T} X_t$$

has an uncertainty distribution

$$\Phi(x) = \inf_{0\le t\le T} \Phi_t(x).$$

Theorem 2. *(Liu (2010b)) Let X_t be an independent increment process and have a continuous uncertainty distribution $\Phi_t(x)$ at each time t. If f is a strictly increasing function, then the supremum*

$$\sup_{0\le t\le T} f(X_t)$$

has an uncertainty distribution

$$\Psi(x) = \inf_{0\le t\le T} \Phi(f^{-1}(x)).$$

Theorem 3. *(Liu (2010b)) Let X_t be an independent increment process and have a continuous uncertainty distribution $\Psi_t(x)$ at each time t. If f is a strictly decreasing function, then the supremum*

$$\sup_{0\le t\le T} f(X_t)$$

has an uncertainty distribution

$$\Psi(x) = 1 - \sup_{0\le t\le s} \Phi_t(f^{-1}(x)).$$

Definition 2. *(Liu (2009a)) An uncertain process C_t is said to be a canonical process if*

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,

(ii) C_t has stationary and independent increments,

(iii) every increment $C_{t+s} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2 whose uncertainty distribution is

$$\Phi(x) = \left(1 + \exp\left(-\frac{p\,x}{\sqrt{3}t}\right)\right)^{-1},\ x \in \Re.$$

If C_t is canonical process, then the uncertain process $X_t = \exp(et + s\,C_t)$ is called a geometric canonical process. Based on the canonical process, the uncertain calculus is introduced as follows.

Definition 3. *(Liu (2009a)) Let X_t be an uncertain process and let C_t be a canonical process. For any partition of closed interval $[a,b]$ with $a = t_1 < t_2 < \mathbf{K} < t_{m+1} = b$, the mesh is written as*

$$\Delta = \max_{1\le i\le k} |t_{i+1} - t_i|.$$

Then the uncertain integral of X_t with respect to C_t is

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta\to 0} \sum_{i=1}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i})$$

provided that the limit exists almost surely and is an uncertain variable.

Let $h(t,c)$ be a continuously differentiable function. Then the uncertain process $X_t = h(t, C_t)$ is an uncertain process. Liu [9] proved the following chain rule

$$\mathrm{d}X_t = \frac{\partial h}{\partial t}(t, C_t)\mathrm{d}t + \frac{\partial h}{\partial c}(t, C_t)\mathrm{d}C_t.$$

An assumption that the stock price follows geometric canonical process was presented by Liu \cite{Liu Uncertain}. In Liu's stock model, the bond price X_t and the stock price Y_t are determined by

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}Y_t = eX_t\mathrm{d}t + s\ X_t\mathrm{d}C_t \end{cases} \tag{1}$$

where r is the riskless interest rate, e is the stock drift, s is the stock diffusion, and C_t is a canonical process. Option pricing problem is a fundamental problem in financial market. European option is the most classic and useful option. A European call option is a contract that gives the holder the right to buy a stock at an expiration time s for a strike price K . Liu (2009a) proposed the European option pricing formulae for Liu's stock model.

3. AMERICAN CALL OPTION PRICE

An American call option is a contract that gives the holder the right to buy a stock at any time prior to an expiration time T for a strike price K. Consider Liu's stock model, we assume that an American call option has strike price K and expiration time T . If Y_t is the price of the underlying stock, then it is clear that the payoff from an American call option is the supremum of $(Y_t - K)^+$ over the time interval $[0,T]$, i.e.,

$$\sup_{0\le t\le T} \exp(-rt)(Y_t - K)^+. \tag{2}$$

Hence the American call option price should be the expected present value of the payoff. Then this option has price

$$f_c = E\left[\sup_{0\le t\le T} \exp(-rt)(Y_t - K)^+\right]. \tag{3}$$

In order to get this American call option price of Liu's stock model, we need to solve the equation (3) in which $Y_t = Y_0\exp(et + s\,C_t)$. Before doing this, we will firstly calculate the uncertainty distribution $\Psi(x)$ of

$$\sup_{0\le t\le T} \exp(-rt)(Y_0\exp(et + s\,C_t) - K)^+.$$

For each $t\in(0,T]$, it is obvious that $\Phi_t(x) = 0$ when $x<0$. If $x\ge 0$, we have

$$\begin{aligned}\Phi_t(x) &= \mathrm{M}\left\{\exp(-rt)(Y_0\exp(et+s\,C_t)-K)^+ \le x\right\}\\ &= \mathrm{M}\left\{Y_0\exp(et+s\,C_t)\le K + x\exp(rt)\right\}\\ &= \mathrm{M}\left\{C_t \le \frac{1}{s}\ln\frac{K+x\exp(rt)}{Y_0} - \frac{et}{s}\right\}\\ &= \left(1+\exp\left(\frac{pe}{\sqrt{3}s} + \frac{p}{\sqrt{3}s\,t}\ln\frac{Y_0}{K+x\exp(rt)}\right)\right)^{-1}.\end{aligned}$$

In order to calculate the uncertainty distribution of

$$\sup_{0\le t\le T} \exp(-rt)(Y_0\exp(et + s\,C_t) - K)^+,$$

We will use the extreme value theorem.

It is obvious that $\exp(-rt)(Y_0\exp(et+s\,C_t)-K)^+$ is an increasing function of independent increment process $et + s\,C_t$ and the uncertainty distribution $\Phi_t(x)$ is continuous for each fixed $t\in(0,T]$. By Theorem 2, the uncertainty distribution $\Psi(x)$ is

$$\begin{aligned}\Psi(x) &= \inf_{0\le t\le T}\Phi_t(x)\\ &= \inf_{0\le t\le T}\left(1+\exp\left(\frac{pe}{\sqrt{3}s} + \frac{p}{\sqrt{3}s\,t}\ln\frac{Y_0}{K+x\exp(rt)}\right)\right)^{-1}\\ &= \left(1+\exp\left(\frac{pe}{\sqrt{3}s} + \frac{p}{\sqrt{3}s\,T}\ln\frac{Y_0}{K+x\exp(rT)}\right)\right)^{-1}.\end{aligned}$$

Theorem 4.
Assume an American call option for Liu's stock model (1) has a strike price K and an expiration time s. Then the American call option price is

$$f_c = \exp(-rT)Y_0\int_{K/Y_0}^{+\infty}\left(1+\exp\left(\frac{p(\ln y - eT)}{\sqrt{3}s\,T}\right)\right)^{-1}\mathrm{d}y.$$

Proof.
By the definition of expected value of uncertain variable, we have

$$f_c = E\left[\sup_{0\le t\le T}\exp(-rt)(Y_0\exp(et+s\,C_t)-K)^+\right].$$

Since $\Psi(x)$ is the uncertainty distribution of

$$\sup_{0\le t\le T}\exp(-rt)(Y_0\exp(et+s\,C_t)-K)^+,$$

we have

$$f_c = \int_0^{+\infty} (1-\Psi(x))\mathrm{d}x.$$

After simplifying and consolidating, we get

$$f_c = \exp(-rT)Y_0 \int_{K/Y_0}^{+\infty} \left(1+\exp\left(\frac{p(\ln y - eT)}{\sqrt{3}s\,T}\right)\right)^{-1} \mathrm{d}y.$$

Theorem 5.
American call option formula of Liu's stock model (1) $f_c = f(Y_0, K, e, s, r, T)$ *has the following properties:*
(i). f_c *is an increasing and convex function of* Y_0*;*
(ii). f_c *is a decreasing and convex function of* K*;*
(iii). f_c *is an increasing function of* e*;*
(iv). f_c *is an increasing function of* s*;*
(v). f_c *is an increasing function of* T*;*
(vi). f_c *is a decreasing function of* r*.*

Proof.
(i). If the other parameters are unchanged, the function

$$\sup_{0\le t\le T} \exp(-rt)(Y_0 \exp(et + s\,C_t) - K)^+$$

is an increasing and convex function of Y_0 and the uncertainty distribution of $\exp(et + s\,C_t)$ is independent of Y_0, therefore f is an increasing and convex function of Y_0.
(ii) It is obvious that

$$\sup_{0\le t\le T} \exp(-rt)(Y_0 \exp(et + s\,C_t) - K)^+$$

is decreasing and convex with respected to K.
(iii) In the equation (3), it is obvious that

$$\left(1+\exp\left(\frac{p(\ln y - eT)}{\sqrt{3}s\,T}\right)\right)^{-1}$$

is an increasing function with respected to e. It means that f_c is increasing with respected to e.
(iv) It is obvious that

$$\left(1+\exp\left(\frac{p(\ln y - eT)}{\sqrt{3}s\,T}\right)\right)^{-1}$$

is increasing of s. Thus the American call option price is increasing with respected to s.
(v) It is easily to see that

$$f_c = E\left[\sup_{0\le t\le T} \exp(-rt)(Y_0 \exp(et + s\,C_t) - K)^+\right]$$

is increasing with respect to T.
(vi) Since $\exp(-rt)$ is decreasing of r, the European call option price is decreasing of r.

Example 1.
Suppose that a stock is presently selling for a price of $Y_0 = 40$, *the riskless interest rate* r *is* 8% *per annum, the stock drift e is 0.06 and the stock diffusion* s *is 0.25. We would like to find an American call option price that expires in three mouths and has a strike price of* $K = 42$ *is about 22.8 cents.*

4. AMERICAN PUT OPTION PRICE

An American put option is a contract that gives the holder the right to sell a stock at any time prior to an expiration time T for a strike price K. Suppose that there is an American put option with strike price K and expiration T in Liu's stock model. If Y_t is the price of the underlying stock, then it is clear that the payoff from an American put option is the supremum of $(K - Y_t)^+$ over the time interval [0, s], i.e.,

$$\sup_{0\le t\le T} \exp(-rt)(K - Y_t)^+$$

Hence the American put option price should be the expected present value of the payoff.

Definition 4. *Assume an American put option has a strike price K and an expiration time T. Then this option has the price*

$$f_p = E\left[\sup_{0\le t\le T} \exp(-rt)(K-Y_t)^+\right]. \tag{4}$$

In order to get this American option price of Liu's stock model, we need to solve the equation (4) in which $Y_t = Y_0 \exp(et + \sigma C_t)$. Before doing this, we will firstly calculate the uncertainty distribution $\Psi(x)$ of

$$\sup_{0\le t\le T} \exp(-rt)(K-Y_t)^+.$$

For each $t \in (0,T]$ and $x < K\exp(-rt)$, the uncertainty distribution is

$$\begin{aligned}\Phi_t(x) &= \mathcal{M}\left\{\exp(-rt)(K - Y_0\exp(et+\sigma C_t))^+ \le x\right\}\\ &= 1 - \mathcal{M}\{Y_0\exp(et+\sigma C_t) < K - x\exp(rt)\}\\ &= \left(1+\exp\left(-\frac{\pi e}{\sqrt{3}\sigma} - \frac{\pi}{\sqrt{3}\sigma t}\ln\frac{Y_0}{K-x\exp(rt)}\right)\right)^{-1}.\end{aligned}$$

In order to calculate the uncertainty distribution of

$$\sup_{0\le t\le T} \exp(-rt)(K-Y_t)^+ ,$$

we need to use extreme value theorem.

By Theorem 3, the uncertainty distribution $\Psi(x)$ of

$$\sup_{0\le t\le T} \exp(-rt)(K-Y_t)^+$$

is

$$\begin{aligned}\Psi(x) &= 1 - \sup_{0\le t\le T}\left(1+\exp\left(\frac{\pi e}{\sqrt{3}\sigma} + \frac{\pi}{\sqrt{3}\sigma t}\ln\frac{Y_0}{K-x\exp(rt)}\right)\right)^{-1}\\ &= \inf_{0\le t\le T}\left(1+\exp\left(-\frac{\pi e}{\sqrt{3}\sigma} - \frac{\pi}{\sqrt{3}\sigma t}\ln\frac{Y_0}{(K-x\exp(rt))\vee 0}\right)\right)^{-1}.\end{aligned}$$

Theorem 6.

Assume an American put option for Liu's stock model (1) has a strike price K and an expiration time T. Then the American put option price is

$$f_p = \int_0^K (1-\Psi(x))\mathrm{d}x$$

where

$$\Psi(x) = \inf_{0\le t\le T}\left(1+\exp\left(-\frac{e\pi}{\sqrt{3}\sigma} - \frac{\pi}{\sqrt{3}\sigma t}\ln\frac{Y_0}{(K-x\exp(rt))\vee 0}\right)\right)^{-1}.$$

Proof.

It follows from the definition of expected value of uncertain variables that

$$\begin{aligned}f_p &= E\left[\sup_{0\le t\le T}\exp(-rt)(K-Y_0\exp(et+\sigma C_t))^+\right]\\ &= \int_0^K (1-\Psi(x))\mathrm{d}x\\ &= \int_0^K \sup_{0\le t\le T}\left(1+\exp\left(\frac{\pi e}{\sqrt{3}\sigma} + \frac{\pi}{\sqrt{3}\sigma t}\ln\frac{Y_0}{(K-x\exp(rt))\vee 0}\right)\right)^{-1}\mathrm{d}x.\end{aligned}$$

Theorem 7.

American put option formula of Liu's stock model

(1) $f_p = f(Y_0, K, e, \sigma, r, T)$ has the following properties:

(i). f_p is a decreasing and convex function of Y_0 ;

(ii). f_p is an increasing and convex function of K ;

(iii). f_p is a decreasing function of e ;

(iv). f_p is an increasing function of σ ;

(v). f_p is a decreasing function of r .

(v). f_p is an increasing function of T .

Proof.

(i). If the other parameters are unchanged, the function

$$\sup_{0\le t\le T} \exp(-rt)(K - Y_0 \exp(et + s\, C_t))^+$$

is a decreasing and convex function of Y_0 and the uncertainty distribution of $\exp(et + s\, C_t)$ is independent of Y_0. Therefore f_p is a decreasing and convex function of Y_0.

(ii) If the other parameters are unchanged, the function

$$\sup_{0\le t\le T} \exp(-rt)(K - Y_0 \exp(et + s\, C_t))^+$$

is an increasing and convex function of K

(iii) In the equation (4), it is obvious that

$$\sup_{0\le t\le T}\left(1+\exp\left(\frac{p e}{\sqrt{3}s}+\frac{p}{\sqrt{3}s\, t}\ln\frac{Y_0}{(K - x\exp(rt))\vee 0}\right)\right)^{-1}$$

is a decreasing function with respected to e. It means that f_p is decreasing with respect to e.

(iv) It is obvious that

$$\sup_{0\le t\le T}\left(1+\exp\left(\frac{p e}{\sqrt{3}s}+\frac{p}{\sqrt{3}s\, t}\ln\frac{Y_0}{(K - x\exp(rt))\vee 0}\right)\right)^{-1}$$

is increasing with respected to s. Thus the American put price is increasing of s.

(v) Since $\exp(-rt)$ is decreasing of r, the American call price is decreasing with respected to r.

(vi) It is easily to see that

$$f_c = E\left[\sup_{0\le t\le T} \exp(-rt)(Y_0 \exp(et + s\, C_t) - K)^+\right]$$

is increasing with respected to T.

Example 2.
Suppose that a stock is presently selling for a price of $Y_0 = 40$, the riskless interest rate r is 8% per annum, the stock drift e is 0.06 and the stock diffusion s is 0.25. We would like to find an American put option price that expires in three months and has a strike price of $K = 38$ is about 3.8 cents.

5. CONCLUSION

In this paper, we investigated the option pricing problems for uncertain financial market. American call and put option price formulas were calculated for Liu's stock model. Some mathematical properties of these formulas were studied.

ACKNOWLEDGEMENTS

This work was supported by National Natural Science Foundation of China Grant No.60874067.

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