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Fuzzy Set Approach to Solve Multi-objective Linear plus Fractional Programming Problem

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Abstract—In this paper we upgraded the Luhandjula's method (Fuzzy Sets and Systems, 13(1), (1984), 11-23) for multiple objective linear plus fractional programming problems (MOL+FPP) with modification given by Dutta *et al.* (Fuzzy Sets and System, 52(1), (1992), 39-45). The aim of this paper is to show new fuzzy set approach for MOL+FPP by defining new membership function for linear function part and similar modified membership function of the goal induced by the quotient part of the objective functions and choose weights corresponding to these goal membership functions. We also provide conditions on the weights indicating the relative importance given by decision maker, so that certain hypothesis verified. We extend the current proof of theorem for MOL+FPP and prove its validity in obtaining the efficient solution.

Keywords—Fuzzy mathematical programming, multiple objective linear plus fractional programming, linguistic variable, membership function.

1. INTRODUCTION

Mathematically, linear fractional programming problem involves optimization of objective function in the form of linear fractional functions *i.e* objective function in the form $N(X)/D(X)$. But in practice fractional programming deals with situation where a relation between physical and / or economical functions. For example cost / time, cost / profit or other quantities that measure the efficiency of a system, is minimized. The state of art in the theory, methods and applications of fractional programming is presented in Stancu Minasian's book (1997). Fractional programming has been widely reviewed by many authors Craven (1998), Horst and Pardalos (1995), Cabellero and Hernadez (2004). But many economical and physical problems of fractional programming involves linear function with addition of quotient function i.e. a new type of optimization problems where the objective function is of the form $L(X)+\dfrac{N(X)}{D(X)}$, subject to certain conditions.

In real world decision situation, decision makers sometimes may face up with the decision to optimize Profit + Inventory/Sales, Salary + Output/Employee etc. with respect to some constraints. Such types of problems with multiple objectives formulate the multiobjective linear- plus-linear fractional programming (MOL+FP) problems. Mathematically, Multiobjective linear-plus-linear fractional programming (MO+FP) problem seeks to optimize more than one objective function in the form of $f(X)+\dfrac{g(X)}{h(X)}$ *i.e* sum of linear function and ratio of two linear functions of non negative variables subject to linear constraints under the assumption that the set of feasible solutions is a convex polyhedral with a finite number of extreme points and the denominator part of each objective function is non zero on the constraint set. In literature, Hirche (1996) investigated the facts about behavior of linear-plus linear fractional objective functions. Recently, Jain and Lachhwani (2009) developed an algorithm to solve multiobjective linear plus fractional program by converting it into fuzzy programming problem.

In the case when several objective functions (conflicting and non commensurable) exists, the optimal solution for a function is not necessarily optimal for the other functions, and hence one introduce the notion of the best compromise solution, also known as Non dominated solution, efficient solution, Non-inferior solution, Pareto's optimal solution. Multiobjective programming problems involve the modeling of input data which can also be made by means of the fuzzy set theory. Significant contributions have been made to fuzzy multi-objective fractional programming problem. For an

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extensive account on fuzzy on fuzzy fractional programming problem with a single or multiple objective functions, we can see Stancu- Minasian and Pop' s review book (2000), Stancu-Minasian (1997), Stancu- Minasian and Pop (2003) etc.. In recent past, hybrids of the stochastic approach and fuzzy approach have been developed. For instance, Wang and Qiao (1993) considered mathematical programming problems with fuzzy random variables.
This paper deals with multiobjective linear plus fractional programming problem (MOL+FPP) i.e.

$$\text{Maximize}\left\{ Z(\mathrm{x}) = (L_1(\mathrm{x}) + \frac{N_1(\mathrm{x})}{D_1(\mathrm{x})}, L_2(\mathrm{x}) + \frac{N_2(\mathrm{x})}{D_2(\mathrm{x})}, ..., L_p(\mathrm{x}) + \frac{N_p(\mathrm{x})}{D_p(\mathrm{x})})\ \mathrm{x} \in \mathrm{X} \right\} \tag{1}$$

Where (i) $X = \left\{ \mathrm{x} \in R^n \,\middle|\, A\mathrm{x} \le b, x \ge 0 \right\}$ is a convex and bounded set.

(ii) A is an $m \times n$ constraint matrix, X is an n- dimensional vector of decision variable and $b \in R^m$.

(iii) $p \ge 2$

(iv) $L_i(\mathrm{x}) = (l^i)' x + \alpha_i$, $N_i(x) = (c^i)' + d_i$, $D_i(x) = (e^i)' x + f_i$, $\forall i = 1, 2, ...p$

(v) $l^i, c^i, e^i \in R^n$, $\alpha_i, d_i, f_i \in R$, $\forall i = 1, 2, ..., p$

(vi) $(e^i)x + f_i > 0 \ \ \forall i = 1, 2, ..., p \ \ \forall x \in X$

The term "Maximize" being used in problem (1) is for finding all weakly and strongly efficient solutions in a maximization sense in terms of following definitions.

Definition 1.1 A point $x^* \in X$ is said to be weakly efficient for problem (1) if and only if there is no $\mathrm{x} \in X$ such that

$$L_i(\mathrm{x}) + \frac{N_i(\mathrm{x})}{D_i(\mathrm{x})} > L_i(\mathrm{x}^*) + \frac{N_i(\mathrm{x}^*)}{D_i(\mathrm{x}^*)} \ \ \forall i = 1, 2, ..., p$$

Definition 1.2 A point $\mathrm{x}^* \in X$ is said to be strongly efficient solution for problem (1) if and only if there is no $\mathrm{x} \in X$ such that

$$L_i(\mathrm{x}) + \frac{N_i(\mathrm{x})}{D_i(\mathrm{x})} \ge L_i(\mathrm{x}^*) + \frac{N_i(\mathrm{x}^*)}{D_i(\mathrm{x}^*)} \ \ \forall i = 1, 2, ..., p$$

and

$$L_{r_o}(\mathrm{x}) + \frac{N_{r_o}(\mathrm{x})}{D_{r_o}(\mathrm{x})} > L_{r_o}(\mathrm{x}^*) + \frac{N_{r_o}(\mathrm{x}^*)}{D_{r_o}(\mathrm{x}^*)}$$

for at least a r_o.
Luhandjula (1984) used a linguistic approach to solve MOL+FPP. Dutta et al (1992) modified the linguistic approach of Luhandjula such as to obtain efficient solution of MOL+FPP. Then Stancu- Minasian and Pop (2003) pointed out certain shortcoming in the work of Dutta *et al.* (1992) and gave correct proof of theorem which validates the obtaining of the efficient solutions under certain hypothesis.
The aim of this paper is to show a new fuzzy set approach for MOL+FPP by defining new membership function for linear function part and similar modified membership function for the goal induced by the quotient part of the objective function and choose weights corresponding to these goal membership functions respectively. We also provide conditions on weights indicating the relative importance of linear part and quotient part of the objective functions given by decision maker so that certain hypothesis verified. It can also be noticed that the method presented as a general one does only work efficiently if certain hypothesis are satisfied.

The paper is organized as follows: In section 2, we propose method to solve MOL+FPP with correct proof of the theorem attesting that, as a result of applying fuzzy method, an efficient point is a solution of problem (1). We also provide conditions on weights in this section. In section 3, we consider an example which illustrates our proposed method. In section 4, we give a comparative analysis of proposed methodology with earlier different approach with considered numerical example.

2. THE PROPOSED METHODOLOGY

In order to propose solution methodology of MOL+FPP, we define the goal membership functions beginning with the concept of (Z, ε) proximity used in the larger frame work of the linguistic variable domain as:

$$C^{N_i}(\mathrm{x}) = \begin{cases} 0 & if\ N_i(x) < p_i \\ \dfrac{N_i(x) - p_i}{N_i^0 - p_i} & if\ p_i \le N_i(x) \le N_i^0 \quad \forall i = 1, 2, ..., p \\ 1 & if\ N_i(x) > N_i^0 \end{cases} \tag{2}$$

$$C^{D_i}(\mathrm{x}) = \begin{cases} 0 & if\ D_i(x) > s_i \\ \dfrac{s_i - D_i(x)}{s_i - D_i^0} & if\ D_i^0 \le D_i(x) \le s_i \quad \forall i = 1, 2, ..., p \\ 1 & if\ D_i(x) < D_i^0 \end{cases} \tag{3}$$

and

$$C^{L_i}(\mathrm{x}) = \begin{cases} 0 & if\ L_i(x) < l_i \\ \dfrac{L_i(x) - l_i}{L_i^0 - l_i} & if\ l_i \le L_i(x) \le L_i^0 \quad \forall i = 1, 2, ..., p \\ 1 & if\ L_i(x) > L_i^0 \end{cases} \tag{4}$$

Where L_i^0, N_i^0 and D_i^0 $(\forall i = 1, 2, ..., p)$ represents the maximal value of linear function $L_i(\mathrm{x})$ and numerator $N_i(\mathrm{x})$ and the minimal value of denominator $D_i(\mathrm{x})$ on the set X respectively where p_i, s_i, l_i are the thresholds beginning with which values of $N_i(\mathrm{x})$, $D_i(\mathrm{x})$ and $L_i(\mathrm{x})$ are acceptable.

Here choice of goal membership functions is motivated by Klir and Yuan (1995) with the following reasons:

1. Since in practice, it is not convenient to calculate the threshold values of each $L_i(x) + \dfrac{N_i(x)}{D_i(x)}$ $i = 1, 2, ..., p$ as it depend on the threshold values of $N_i(\mathrm{x})$, $D_i(\mathrm{x})$ and $L_i(\mathrm{x})$ separately. Therefore its individual threshold values of $N_i(\mathrm{x})$, $D_i(\mathrm{x})$ and $L_i(\mathrm{x})$ are considered.
2. Acceptable threshold values for $N_i(\mathrm{x})$, $L_i(\mathrm{x})$ are obtained by considering their respective maximum values and for $D_i(\mathrm{x})$ by their minimum values in the constraint region.

As a membership function of the goal *i* induced by the objective function $L_i(\mathrm{x}) + \dfrac{N_i(\mathrm{x})}{D_i(\mathrm{x})}$, we choose the function

$$\mu_i(\mathrm{x}) = w_i \min\{C^{L_i}(x), C^{N_i}(x)\} + w_i' C^{D_i}(x) \qquad i = 1, 2, ..., p$$

Where w_i an w_i' are the weights indicating the relative importance given by decision maker to the criteria and verifying the condition $\sum_{i=1}^{p}\left(w_i + w_i'\right) = 1$

Conditions on weights

We emphasize that the membership function $\mu_i(x)\ \forall\, i = 1, 2, ..., p$ used in the following verify the hypothesis: $\forall\ \mathrm{x}^1, \mathrm{x}^2 \in X$ if

$$L_i(\mathrm{x}^1) + \frac{N_i(\mathrm{x}^1)}{D_i(\mathrm{x}^1)} > L_i(\mathrm{x}^2) + \frac{N_i(\mathrm{x}^2)}{D_i(\mathrm{x}^2)}$$

then

$$\mu_i(\mathrm{x}^1) > \mu_i(\mathrm{x}^2) \qquad \forall i = 1, 2, ..., p \tag{5}$$

Hypothesis (5) is used, however, to prove the efficiency of the solution obtained by solving the problem

$$\max\ V(\mu) = \sum_{i=1}^{p}\left\{w_i \mu_i^{NL} + w_i' \mu_i^{D_i}\right\}$$

$$\text{Subject to } \mu_i^{NL} = \min\left\{C^{N_i}(\mathrm{x}), C^{L_i}(\mathrm{x})\right\},\ \mu_i^{D_i} = C^{D_i}(\mathrm{x}),$$

$$0 \le \mu_i^{NL} \le 1,\ 0 \le \mu_i^{D_i} \le 1,\ \forall i = 1, 2, ..., p,$$

$$Ax \le b,\ x \ge 0 \quad \text{and} \quad \sum_{i=1}^{p}\left(w_i + w_i'\right) = 1 \tag{6}$$

Now we provide conditions on the weights w_i, w_i' so that $\mu_i(\mathrm{x})$ verifies the hypothesis.

$$(\forall\ \mathrm{x}^1, \mathrm{x}^2 \in X) \qquad \left(L_i(\mathrm{x}^1) + \frac{N_i(\mathrm{x}^1)}{D_i(\mathrm{x}^1)} > L_i(\mathrm{x}^2) + \frac{N_i(\mathrm{x}^2)}{D_i(\mathrm{x}^2)} \Rightarrow \mu_i(\mathrm{x}^1) < \mu_i(\mathrm{x}^2)\right)$$

So, $\mu_i(\mathrm{x}^1) < \mu_i(\mathrm{x}^2) \Leftrightarrow w_i \min\left\{C^{N_i}(\mathrm{x}^1), C^{L_i}(\mathrm{x}^1)\right\} + w_i' C_i^{D_i}(\mathrm{x}^1) < w_i \min\left\{C^{N_i}(\mathrm{x}^2), C^{L_i}(\mathrm{x}^2)\right\} + w_i' C_i^{D_i}(\mathrm{x}^2)$

$$\Leftrightarrow w_i\left[\min\left\{C^{N_i}(\mathrm{x}^1), C^{L_i}(\mathrm{x}^1)\right\} - \min\left\{C^{N_i}(\mathrm{x}^2), C^{L_i}(\mathrm{x}^2)\right\}\right] < w_i^{'}\left[C_i^{D_i}(\mathrm{x}^2) - C_i^{D_i}(\mathrm{x}^1)\right] \tag{7}$$

Case I. when $\min\left\{C^{N_i}(\mathrm{x}^1), C^{L_i}(\mathrm{x}^1)\right\} = C^{N_i}(\mathrm{x}^1)$ and $\min\left\{C^{N_i}(\mathrm{x}^2), C^{L_i}(\mathrm{x}^2)\right\} = C^{N_i}(\mathrm{x}^2)$

So, $\mu_i(\mathrm{x}^1) < \mu_i(\mathrm{x}^2)$

$$\Leftrightarrow w_i C^{N_i}(\mathrm{x}^1) + w_i^{'} C^{D_i}(\mathrm{x}^1) < w_i C^{N_i}(\mathrm{x}^2) + w_i^{'} C^{D_i}(\mathrm{x}^2)$$

$$\Leftrightarrow w_i\left[C^{N_i}(\mathrm{x}^1) - C^{N_i}(\mathrm{x}^2)\right] < w_i^{'}\left[C^{D_i}(\mathrm{x}^2) - C^{D_i}(\mathrm{x}^1)\right]$$

$$\Leftrightarrow w_i\left[\frac{N_i(\mathrm{x}^1) - p_i}{N_i^0 - p_i} - \frac{N_i(\mathrm{x}^2) - p_i}{N_i^0 - p_i}\right] < w_i^{'}\left[\frac{s_i - D_i(\mathrm{x}^2)}{s_i - D_i^0} - \frac{s_i - D_i(\mathrm{x}^1)}{s_i - D_i^0}\right]$$

$$\Leftrightarrow w_i\left[\frac{N_i(\mathrm{x}^1) - N_i(\mathrm{x}^2)}{N_i^0 - p_i}\right] < w_i^{'}\left[\frac{D_i(\mathrm{x}^1) - D_i(\mathrm{x}^2)}{s_i - D_i^0}\right]$$

Or, putting $k_i = \dfrac{s_i - D_i^0}{N_i^0 - p_i}$

We obtain $\dfrac{w_i^{'}}{w_i} > k_i\left[\dfrac{N_i(\mathrm{x}^1) - N_i(\mathrm{x}^2)}{D_i(\mathrm{x}^1) - D_i(\mathrm{x}^2)}\right]$ iff $D_i(\mathrm{x}^1) > D_i(\mathrm{x}^2)$

and $\dfrac{w_i^{'}}{w_i} < k_i\left[\dfrac{N_i(\mathrm{x}^1) - N_i(\mathrm{x}^2)}{D_i(\mathrm{x}^1) - D_i(\mathrm{x}^2)}\right]$ iff $D_i(\mathrm{x}^1) < D_i(\mathrm{x}^2)$

It follows that $w_i^{'}/w_i < k_i \overline{A_i}$ and $w_i^{'}/w_i > k_i \underline{A_i}$ where

$$\overline{A_i} = \min\left\{\frac{N_i(\mathrm{x}^1) - N_i(\mathrm{x}^2)}{D_i(\mathrm{x}^1) - D_i(\mathrm{x}^2)}\middle| D_i(\mathrm{x}^1) < D_i(\mathrm{x}^2), \frac{N_i(\mathrm{x}^1)}{D_i(\mathrm{x}^1)} < \frac{N_i(\mathrm{x}^2)}{D_i(\mathrm{x}^2)}, \mathrm{x}^1, \mathrm{x}^2 \in X\right\}$$

and

$$\underline{A_i} = \max\left\{\frac{N_i(\mathrm{x}^1) - N_i(\mathrm{x}^2)}{D_i(\mathrm{x}^1) - D_i(\mathrm{x}^2)}\middle| D_i(\mathrm{x}^1) > D_i(\mathrm{x}^2), \frac{N_i(\mathrm{x}^1)}{D_i(\mathrm{x}^1)} < \frac{N_i(\mathrm{x}^2)}{D_i(\mathrm{x}^2)}, \mathrm{x}^1, \mathrm{x}^2 \in X\right\}$$

Thus, if $k_i \underline{A_i} < w_i^{'}/w_i < k_i \overline{A_i}$ then, hypothesis (5) is verified. This is same as given by Stancu- Minasian and Pop (2003).

Case II. When $\min\left\{C^{N_i}(\mathrm{x}^1), C^{L_i}(\mathrm{x}^1)\right\} = C^{L_i}(\mathrm{x}^1)$ and $\min\left\{C^{N_i}(\mathrm{x}^2), C^{L_i}(\mathrm{x}^2)\right\} = C^{L_i}(\mathrm{x}^2)$ So, $\mu_i(\mathrm{x}^1) < \mu_i(\mathrm{x}^2)$

$$\Leftrightarrow w_i C^{L_i}(\mathrm{x}^1) + w_i^{'} C^{D_i}(\mathrm{x}^1) < w_i C^{L_i}(\mathrm{x}^2) + w_i^{'} C^{D_i}(\mathrm{x}^2)$$

$$\Leftrightarrow w_i\left[C^{L_i}(\mathrm{x}^1) - C^{L_i}(\mathrm{x}^2)\right] < w_i^{'}\left[C^{D_i}(\mathrm{x}^2) - C^{D_i}(\mathrm{x}^1)\right]$$

$$\Leftrightarrow w_i\left[\frac{L_i(\mathrm{x}^1) - l_i}{L_i^0 - l_i} - \frac{L_i(\mathrm{x}^2) - l_i}{L_i^0 - l_i}\right] < w_i^{'}\left[\frac{s_i - D_i(\mathrm{x}^2)}{s_i - D_i^0} - \frac{s_i - D_i(\mathrm{x}^1)}{s_i - D_i^0}\right]$$

$$\Leftrightarrow w_i\left[\frac{L_i(\mathrm{x}^1) - L_i(\mathrm{x}^2)}{L_i^0 - l_i}\right] < w_i^{'}\left[\frac{D_i(\mathrm{x}^1) - D_i(\mathrm{x}^2)}{s_i - D_i^0}\right]$$

Or, putting $k_i^{'} = \dfrac{s_i - D_i^0}{L_i^0 - l_i}$

We obtain $\dfrac{w_i^{'}}{w_i} > k_i^{'}\left[\dfrac{L_i(\mathrm{x}^1) - L_i(\mathrm{x}^2)}{D_i(\mathrm{x}^1) - D_i(\mathrm{x}^2)}\right]$ iff $D_i(\mathrm{x}^1) > D_i(\mathrm{x}^2)$

and $\dfrac{w_i^{'}}{w_i} < k_i^{'}\left[\dfrac{L_i(\mathrm{x}^1) - L_i(\mathrm{x}^2)}{D_i(\mathrm{x}^1) - D_i(\mathrm{x}^2)}\right]$ iff $D_i(\mathrm{x}^1) < D_i(\mathrm{x}^2)$

It follows that $w_i^{'}/w_i < k_i^{'} \overline{B_i}$ and $w_i^{'}/w_i > k_i \underline{B_i}$ where

$$\overline{B_i} = \min\left\{\frac{L_i(\mathrm{x}^1) - L_i(\mathrm{x}^2)}{D_i(\mathrm{x}^1) - D_i(\mathrm{x}^2)}\middle| D_i(\mathrm{x}^1) < D_i(\mathrm{x}^2), L_i(\mathrm{x}^1) + \frac{N_i(\mathrm{x}^1)}{D_i(\mathrm{x}^1)} < L_i(\mathrm{x}^2) + \frac{N_i(\mathrm{x}^2)}{D_i(\mathrm{x}^2)}, \mathrm{x}^1, \mathrm{x}^2 \in X\right\}$$

And $$\underline{B_i} = \max\left\{ \frac{L_i(\mathrm{x}^1) - L_i(\mathrm{x}^2)}{D_i(\mathrm{x}^1) - D_i(\mathrm{x}^2)} \middle| D_i(\mathrm{x}^1) > D_i(\mathrm{x}^2), L_i(\mathrm{x}^1) + \frac{N_i(\mathrm{x}^1)}{D_i(\mathrm{x}^1)} < L_i(\mathrm{x}^2) + \frac{N_i(\mathrm{x}^2)}{D_i(\mathrm{x}^2)}, \mathrm{x}^1, \mathrm{x}^2 \in X \right\}$$

Thus, if $k_i^{'} \underline{B_i} < w_i^{'} / w_i < k_i^{'} \overline{B_i}$ then, hypothesis (5) is verified.

Case III. When $\min\{C^{N_i}(\mathrm{x}^1), C^{L_i}(\mathrm{x}^1)\} = C^{N_i}(\mathrm{x}^1)$ and $\min\{C^{N_i}(\mathrm{x}^2), C^{L_i}(\mathrm{x}^2)\} = C^{L_i}(\mathrm{x}^2)$ So, $\mu_i(\mathrm{x}^1) < \mu_i(\mathrm{x}^2)$

$$\Leftrightarrow w_i C^{N_i}(\mathrm{x}^1) + w_i^{'} C^{D_i}(\mathrm{x}^1) < w_i C^{L_i}(\mathrm{x}^2) + w_i^{'} C^{D_i}(\mathrm{x}^2)$$

$$\Leftrightarrow w_i\left[C^{N_i}(\mathrm{x}^1) - C^{L_i}(\mathrm{x}^2)\right] < w_i^{'}\left[C^{D_i}(\mathrm{x}^2) - C^{D_i}(\mathrm{x}^1)\right]$$

$$\Leftrightarrow w_i\left[\frac{N_i(\mathrm{x}^1) - p_i}{N_i^0 - p_i} - \frac{L_i(\mathrm{x}^2) - l_i}{L_i^0 - l_i}\right] < w_i^{'}\left[\frac{s_i - D_i(\mathrm{x}^2)}{s_i - D_i^0} - \frac{s_i - D_i(\mathrm{x}^1)}{s_i - D_i^0}\right]$$

$$\Leftrightarrow w_i\left[\frac{N_i(\mathrm{x}^1) - p_i}{\Delta_{N_i}} - \frac{L_i(\mathrm{x}^2) - l_i}{\Delta_{L_i}}\right] < w_i^{'}\left[\frac{D_i(\mathrm{x}^1) - D_i(\mathrm{x}^2)}{\Delta_{D_i}}\right]$$

$$\Leftrightarrow w_i\left[\frac{\{N_i(\mathrm{x}^1) - p_i\}\Delta_{L_i} - \{L_i(\mathrm{x}^2) - l_i\}\Delta_{N_i}}{\Delta_{N_i}\Delta_{L_i}}\right] < w_i^{'}\left[\frac{D_i(\mathrm{x}^1) - D_i(\mathrm{x}^2)}{\Delta_{D_i}}\right]$$

$$\Leftrightarrow \frac{w_i^{'}}{w_i} > \frac{\Delta_{D_i}}{\Delta_{N_i}\Delta_{L_i}}\left[\frac{\{N_i(\mathrm{x}^1) - p_i\}\Delta_{L_i} - \{L_i(\mathrm{x}^2) - l_i\}\Delta_{N_i}}{D_i(\mathrm{x}^1) - D_i(\mathrm{x}^2)}\right] \quad \text{iff } D_i(\mathrm{x}^1) > D_i(\mathrm{x}^2)$$

and $$\Leftrightarrow \frac{w_i^{'}}{w_i} < \frac{\Delta_{D_i}}{\Delta_{N_i}\Delta_{L_i}}\left[\frac{\{N_i(\mathrm{x}^1) - p_i\}\Delta_{L_i} - \{L_i(\mathrm{x}^2) - l_i\}\Delta_{N_i}}{D_i(\mathrm{x}^1) - D_i(\mathrm{x}^2)}\right] \quad \text{iff } D_i(\mathrm{x}^1) < D_i(\mathrm{x}^2)$$

Or, putting $$k_i^{''} = \frac{\Delta_{D_i}}{\Delta_{N_i}\Delta_{L_i}}$$

We obtain $$\frac{w_i^{'}}{w_i} > k_i^{''}\left[\frac{\{N_i(\mathrm{x}^1) - p_i\}\Delta_{L_i} - \{L_i(\mathrm{x}^2) - l_i\}\Delta_{N_i}}{D_i(\mathrm{x}^1) - D_i(\mathrm{x}^2)}\right] \quad \text{iff } D_i(\mathrm{x}^1) > D_i(\mathrm{x}^2)$$

and $$\frac{w_i^{'}}{w_i} < k_i^{''}\left[\frac{\{N_i(\mathrm{x}^1) - p_i\}\Delta_{L_i} - \{L_i(\mathrm{x}^2) - l_i\}\Delta_{N_i}}{D_i(\mathrm{x}^1) - D_i(\mathrm{x}^2)}\right] \quad \text{iff } D_i(\mathrm{x}^1) < D_i(\mathrm{x}^2)$$

Similarly for the case $\min\{C^{N_i}(\mathrm{x}^1), C^{L_i}(\mathrm{x}^1)\} = C^{L_i}(\mathrm{x}^1)$ and $\min\{C^{N_i}(\mathrm{x}^2), C^{L_i}(\mathrm{x}^2)\} = C^{N_i}(\mathrm{x}^2)$
Condition will be

$$\frac{w_i^{'}}{w_i} > \frac{\Delta_{D_i}}{\Delta_{N_i}\Delta_{L_i}}\left[\frac{\{L_i(\mathrm{x}^1) - l_i\}\Delta_{N_i} - \{N_i(\mathrm{x}^2) - p_i\}\Delta_{L_i}}{D_i(\mathrm{x}^1) - D_i(\mathrm{x}^2)}\right] \quad \text{iff } D_i(\mathrm{x}^1) > D_i(\mathrm{x}^2)$$

$$\frac{w_i^{'}}{w_i} < \frac{\Delta_{D_i}}{\Delta_{N_i}\Delta_{L_i}}\left[\frac{\{L_i(\mathrm{x}^1) - l_i\}\Delta_{N_i} - \{N_i(\mathrm{x}^2) - p_i\}\Delta_{L_i}}{D_i(\mathrm{x}^1) - D_i(\mathrm{x}^2)}\right] \quad \text{iff } D_i(\mathrm{x}^1) < D_i(\mathrm{x}^2)$$

So that hypothesis (5) is verified.
Now we can present the following propositions for MOL+FPP.

Proposition 2.1 Assume that hypothesis (5) holds. If $\mathrm{x}^{\mathrm{opt}}$ is an optimal solution for problem (6), then $\mathrm{x}^{\mathrm{opt}}$ is weakly efficient solution for the problem (1).
Proof: Let $\mathrm{x}^{\mathrm{opt}}$ be optimal solution for problem (6) and assume that $\mathrm{x}^{\mathrm{opt}}$ is not weakly efficient solution for problem (1). Hence, there is a vector x $\in$ X such that

$$L_i(\mathrm{x}) + \frac{N_i(\mathrm{x})}{D_i(\mathrm{x})} > L_i(\mathrm{x}^{\mathrm{opt}}) + \frac{N_i(\mathrm{x}^{\mathrm{opt}})}{D_i(\mathrm{x}^{\mathrm{opt}})} \quad \forall i = 1, 2, \ldots., p$$

From hypothesis (5), it follows that $\mu_i(\mathrm{x}) > \mu_i(\mathrm{x}^{\mathrm{opt}}) \ \forall i = 1, 2, \ldots., p$

$$w_i \min\{C^{N_i}(\mathrm{x}), C^{L_i}(\mathrm{x})\} + w_i^{'} C_i^{D_i}(\mathrm{x}) > w_i \min\{C^{N_i}(\mathrm{x}^{\mathrm{opt}}), C^{L_i}(\mathrm{x}^{\mathrm{opt}})\} + w_i^{'} C_i^{D_i}(\mathrm{x}^{\mathrm{opt}})$$

Thus $$\sum_{i=1}^{p}\{w_i \mu_i^{NL}(\mathrm{x}) + w_i^{'} \mu_i^{D_i}(\mathrm{x})\} > \sum_{i=1}^{p}\{w_i \mu_i^{NL}(\mathrm{x}^{\mathrm{opt}}) + w_i^{'} \mu_i^{D_i}(\mathrm{x}^{\mathrm{opt}})\}$$

which contradict the assumption that x^{opt} is an optimal solution for problem (6). The proof is complete. We can similarly prove even the strong efficiency of solution x^{opt} for problem (1) without the further hypothesis that x^{opt} being a unique solution of problem (6) being necessary, as assumed by Dutta *et al.* (1992).

Proposition 2.2 Assume that hypothesis (5) holds. If x^{opt} is an optimal solution for problem (6), then x^{opt} is strongly efficient solution for problem (1).

Proof: Let x^{opt} be optimal solution for problem (6) and assume that x^{opt} is not strongly efficient solution to problem (1). Hence, a x $\in$ X exists such that

$$L_i(\mathrm{x}) + \frac{N_i(\mathrm{x})}{D_i(\mathrm{x})} \geq L_i(\mathrm{x}^{\mathrm{opt}}) + \frac{N_i(\mathrm{x}^{\mathrm{opt}})}{D_i(\mathrm{x}^{\mathrm{opt}})} \quad \forall i = 1, 2, ..., p$$

And, for at least an index *j*, we have

$$L_i(\mathrm{x}) + \frac{N_i(\mathrm{x})}{D_i(\mathrm{x})} > L_i(\mathrm{x}^{\mathrm{opt}}) + \frac{N_i(\mathrm{x}^{\mathrm{opt}})}{D_i(\mathrm{x}^{\mathrm{opt}})}$$

From hypothesis (5) it results that $\mu_i(\mathrm{x}) \geq \mu_i(\mathrm{x}^{\mathrm{opt}})$ for all $i = 1, 2, ..., p$ and for the index *j*, $\mu_j(\mathrm{x}) > \mu_j(\mathrm{x}^{\mathrm{opt}})$, multiplying these relations by $w_i \geq 0$ and $w_i^{'} \geq 0$ respectively, and summing after all *i*, yields

$$\sum_{i=1}^{p} \left\{ w_i \mu_i^{NL}(\mathrm{x}) + w_i^{'} \mu_i^{D_i}(\mathrm{x}) \right\} \; > \; \sum_{i=1}^{p} \left\{ w_i \mu_i^{NL}(\mathrm{x}^{\mathrm{opt}}) + w_i^{'} \mu_i^{D_i}(\mathrm{x}^{\mathrm{opt}}) \right\}$$

And this is again in contradiction with the optimality of x^{opt} for problem (6).

3. COMPUTATIONAL RESULTS

In this section, we consider the following example to explain our argument.

$$\max \left\{ f_1(\mathrm{x}) = x_1 + \frac{x_1 + x_2 - 1}{-x_1 + 2x_2 + 7}, \; f_2(\mathrm{x}) = -x_2 + \frac{2x_1 + x_2 - 2}{x_2 + 4} \right\}$$

$$\begin{aligned} \text{Subject to} \quad & -x_1 + 3x_2 \leq 0 \\ & x_1 \leq 6 \\ \text{and} \quad & x_1, x_2 \geq 0 \end{aligned} \qquad (9)$$

is approached by identifying the concrete form (6) as:

$$\max \; V(\mu) = \sum_{i=1}^{p} \left\{ w_i \mu_i^{NL} + w_i^{'} \mu_i^{D_i} \right\}$$

$$\begin{aligned} \text{Subject to} \quad & \mu_i^{NL} = \min\left\{ C^{N_i}(\mathrm{x}), C^{L_i}(\mathrm{x}) \right\}, \; \mu_I^{D_i} = C^{D_i}(\mathrm{x}) \\ & 0 \leq \mu_i^{NL} \leq 1, \; 0 \leq \mu_i^{D_i} \leq 1, \forall i = 1, 2, ..., p, \\ & Ax \leq b, \; x \geq 0 \quad \text{and} \quad \sum_{i=1}^{p} \left(w_i + w_i^{'} \right) = 1 \end{aligned} \qquad (10)$$

Here we used weights $w_1 = 0.035, \; w_1^{'} = 0.465, \quad w_2 = 0.475, \; w_2^{'} = 0.025$ so that it satisfies $\sum_{i=1}^{p} \left(w_i + w_i^{'} \right) = 1$.

We assume that $\mu_i^{NL} = \min\left\{ C^{N_i}(\mathrm{x}), C^{L_i}(\mathrm{x}) \right\} = C^{L_i}(\mathrm{x}), \; \forall i = 1, 2$

For *i=1* $\quad \mu_1^{NL} = C^{L_1}(\mathrm{x}) = \frac{x_1}{6}, \; \mu_1^{D} = C^{D_1}(\mathrm{x}) = \frac{x_1 - 2x_2}{6}, \quad \mu_2^{NL} = C^{L_2}(\mathrm{x}) = \frac{-x_2 + 2}{2}, \quad \mu_2^{D} = C^{D_2}(\mathrm{x}) = \frac{2 - x_2}{2}.$

Here the threshold values of individual membership functions are assumed are:

$$l_1 = 0, L_1^0 = 6, \quad l_2 = -2, L_2^0 = 0, \; s_1 = 7, D_1^0 = 1, \quad s_2 = 6, D_2^0 = 4,$$

Substituting these values in the problem (10), we get

$$\max \; V(\mu) = 0.08333x_1 - 0.4175x_2 + 0.50$$

$$\begin{aligned} \text{Subject to} \quad & -x_1 + 3x_2 \leq 0 \\ & x_1 \leq 6 \\ \text{and} \quad & x_1, x_2 \geq 0 \end{aligned}$$

Solving this linear programming problem by simplex method, we get solution $x_1 = 6, \; x_2 = 0$ with $f_1(\mathrm{x}) = 11, \; f_2(\mathrm{x}) = 2.5$. Thus x^{opt} = (6, 0) is the single efficient point of problem (9) because both the objective functions reach in this point their optimum, independently one from another on the same feasible region. This efficient point (6, 0) can be obtained not as a solution of a problem (6) for any choice of the weights ($w_1, w_1^{'}, w_2, w_2^{'}$) but for a choice of ($w_1, w_1^{'}, w_2, w_2^{'}$) from a region included in OXYZ.

The efficient solution depends on the choice of respective weights ($w_1, w_1^{'}, w_2, w_2^{'}$) given to linear - numerator part and denominator part of objective function satisfying the certain conditions described. Here the weights assumed satisfy the conditions as discussed in case II

i.e. $\mu_i^{NL} = \min\left\{C^{N_i}(\mathrm{x}), C^{L_i}(\mathrm{x})\right\} = C^{L_i}(\mathrm{x}), \;\; \forall i = 1, 2$

and also common condition $\sum_{i=1}^{p}\left(w_i + w_i^{'}\right) = 1$. We can easily identify the feasible region of weights *w* for which the efficient point (6, 0) of problem (9) can be obtained as a solution of problem (6). This region is prescribed in figure 1 by the polyhedral set ABCOD. The interior of the polyhedral set XOYZ is the feasible region of values $w_1, w_1^{'}, w_2, w_2^{'}$ because there is a relation $w_1 + w_1^{'} + w_2 + w_2^{'} = 1$ and $w_1, w_1^{'}, w_2, w_2^{'} \geq 0$ between them.

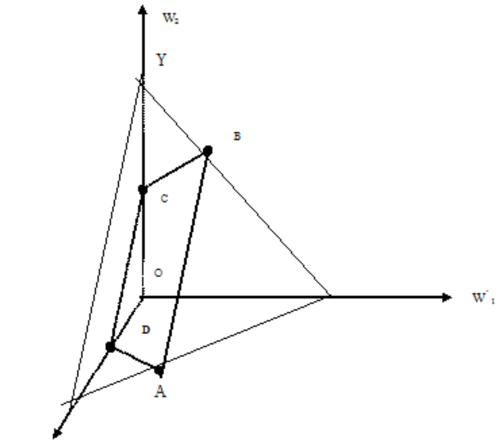

Figure 1. Feasible Region of Weights w

4. COMPARATIVE ANALYSIS

Numerous methods for solving multiobjective linear and linear fractional programming problems have been suggested in the literature. Some of them are Gupta and Chakraborty (1997, 2002) and Jain and Lachhwani (2009) etc. and many other researchers used and modified the concept of multiobjective decision making problems and discussed different approaches to tackle these problems. Here we compare the proposed methodology with the earlier approach given by Jain and Lachhwani (2009) in the context of above numerical example. Using the methodology given by Jain and Lachhwani (2009), the problem (1) can be reduced to

$$\begin{aligned} &\text{Max} \quad \lambda \\ &\text{subject to} \quad -L_i(\mathrm{x})D_i(\mathrm{x}) - N_i(\mathrm{x}) + \overline{Z_i}D_i(\mathrm{x}) \leq (-p\lambda + p)D_i(\mathrm{x}) \qquad i = 1, 2, ..., p \\ &\qquad Ax \leq b \\ &\qquad \mathrm{x}, \lambda \geq 0 \end{aligned} \tag{8}$$

where $\overline{Z_i}$ is the maximum value of $Z_i(X)$, distance function d with unit weight as $d_i(X) = \left|\overline{Z_i} - Z_i(X)\right|$ and $p = \sup\left\{\overline{d_i}\right\}$ $\forall i = 1, 2, 3, ..., k$. Problem (8) is a non-linear programming problem and can be solved by non-linear techniques. Using the above methodology to the given example, the reduced problem will be

$$\begin{aligned} &\max \quad \lambda \\ &\text{subject to,} \quad x_1^2 - 2x_1x_2 - 16.5x_1 + 16x_2 - 2.5x_1\lambda + 5x_2\lambda + 17.5\lambda \leq -60.5 \\ &\qquad x_2^2 - 5x_2 - 2x_1 + 2.5x_2\lambda + 10\lambda \leq -2 \\ &\qquad -x_1 + 3x_2 \leq 0 \\ &\qquad x_1 \leq 6 \\ &\text{and} \qquad x_1, x_2 \geq 0 \end{aligned}$$

Solving it using non linear programming techniques or software package like LINGO 10 (trial version) as shown in figure 2(a) and 2(b), the optimal solution of the problem is obtained as:

IJOR Vol. 8, No. 3, 15−23 (2011)

$x_1 = 6,\ x_2 = 0 \quad \lambda = 1$ with $f_1(\mathrm{x}) = 11,\ f_2(\mathrm{x}) = 2.5$

Which is same as obtained in our proposed approach. This also verifies the proposed approach.

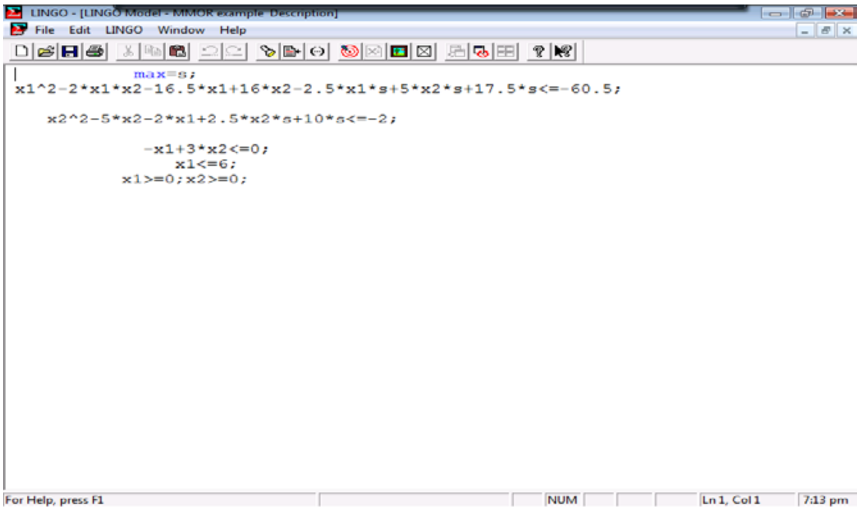

Figure 2 (a): Description of example in LINGO 10 (trial version)

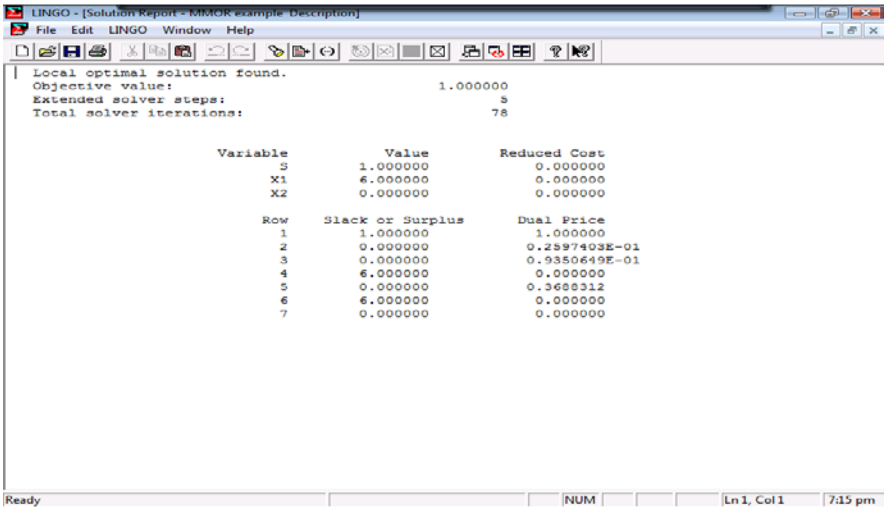

Figure 2(b): Solution of example using LINGO 10 (trial version)

5. SPECIAL CASE

Our proposed methodology for MOL+FPP can be deduced to the Luhandjula's (1984) method for solving multi objective linear fractional programming problem (MOLFPP) by taking linear function part $L_i(\mathrm{x}) = 0, \quad \forall i = 1, 2, \ldots, p$ in each objective function which also justifies our proposed method.

6. CONCLUSION

An effort has been made to describe fuzzy set approach to solve MOL+FPP and provide conditions on weights given by decision maker so that certain hypothesis verified. The proposed methodology can be useful new approach to handle MOL+FPP provided certain hypotheses on weights are verified.

REFERENCES

1. Craven, B. D. (1998). *Fractional programming*, In: sigma series in applied mathematics, 4, Berlin, Heldermann Verlag.
2. Caballero, R. and Hernandez, M. (2004). The controlled estimation method in multiobjective linear fractional problem, *Computers and Operations Research*, 31(11): 1821-1832.
3. Chakraborty, M. and Gupta, S. (1997). Multiobjective linear programming: A fuzzy programming approach, *International Journal of Management and Systems*, 13(2): 207-214.

4. Chakraborty, M. and Gupta, Sandipan. (2002). Fuzzy mathematical programming for multiobjective linear fractional programming problem, *Fuzzy Sets and Systems*, 125(3): 335-342.
5. Dutta, D. Tiwari, R. N. and Rao, R. (1992). Multiple objective linear fractional programming problem- A fuzzy set theoretic approach, *Fuzzy Sets and Systems*, 52(1): 39-45.
6. Horst, R. and Pardalos, P. M. (1995). Editor, *Handbook of global optimization*, Kluwer Academic Publishers, Dordrecht.
7. Hirche, J. (1996). A note on programming problems with linear plus linear fractional objective functions, *European Journal of Operational Research*, 89(1): 212-214.
8. Jain, S. and Lachhwani, K. (2009). An algorithm for the solution of multiobjective linear plus fractional program, Bulletin *of Australian Society of Operations Research*, 28(2): 2-8.
9. Klir, G. J. and Yuan, B. (1995). *Fuzzy sets and fuzzy logic: Theory and applications*, Prentice-Hall, PTR, USA.
10. Luhandjula, M. K. (1984). Fuzzy approaches for multiple objective linear fractional optimization, *Fuzzy Sets and Systems*, 13: 11-23.
11. Stancu- Minasian, I. M. (1997). *Fractional programming, theory, methods and applications*, Kluwer Academic Publishers, Dordrecht, Boston, London.
12. Stancu- Minasian, I. M. and Pop, B. (2000). Fuzzy fractional programming: Some recent results, paper presented at 2nd International Conference on symmetry and antisymmetry in mathematics, Formal languages and Computer sciences, Brasov, Romania.
13. Stancu- Minasian, I. M. and Pop, B. (2003). On a fuzzy set approach to solving multiple objective linear fractional programming problem, *Fuzzy Sets and Systems*, 134: 397-405.
14. Wang, G. Y. and Qiao, Z. (1993). Fuzzy programming with fuzzy random variable coefficients, *Fuzzy Sets and Systems*, 57: 295-311.