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Possibilistic mean- variance- skewness portfolio selection models

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Abstract—In portfolio selection problem, the investor usually makes his portfolio decision according to his experience and his economic acquaintance. So, deterministic portfolio selection is not a good choice for the investor. In most of the recent works on this problem, fuzzy set theory is widely used to model the problem in uncertain environment. Since portfolio returns are in general asymmetric, in addition to the general concept of considering mean and variance, the third moment skewness should be considered for a better decision making. This paper applies the concept of weighted possibilistic moments of fuzzy numbers to extend the classical mean- variance portfolio selection model under the consideration of skewness. Return ratios are considered as fuzzy numbers and its possibilistic moments are evaluated to formulate the models. In order to solve the models, fuzzy simulation (FS) and elitist genetic algorithm (EGA) are integrated to produce more powerful and effective hybrid intelligent algorithm (HIA). Finally, our approaches are tested on a set of stock data from Bombay Stock Exchange (BSE).
Keywords— Investment analysis, portfolio selection, mean-variance-skewness model, fuzzy numbers, weighted possibilistic moments, hybrid intelligent algorithm.

1. INTRODUCTION

Although the foundation of modern mathematical models in economics can be traced back to Bachelier's (1900) dissertation on the theory of speculation, without hesitation, the work of Markowitz (1952) in portfolio selection has been the most impact-making development in mathematical finance. Most of the reasonable works on portfolio selection has been done based on only the first two moments of return distribution. The first order moment about the origin, i.e., the mean, quantifies the return and the second order moment about the mean, i.e., the variance, quantifies the risk. Now the third order moment about the mean of a return distribution i.e., skewness measures the asymmetry of the distribution. A natural extension of the mean- variance model is to add the skewness as a factor for consideration in portfolio management. One interested in considering skewness prefers a portfolio with a higher probability of large payoffs when mean and variance remain same. The importance of higher order moments in portfolio selection was suggested by Samuelson (1958). But consideration of skewness has been started by Lai (1991) and is continued by Konno and Shirakawa (1993), Konno and Suzuki (1995), Chunhachinda *et al.* (1997), Prakash *et al.* (2003), Briec *et al.* (2007), Yu *et al.* (2008) and others.

All the above literatures assume that the security returns are random variables. There are many non-stochastic factors that affect stock markets. Dealing those factors with probability approaches is inappropriate. By incurring fuzzy approaches quantitative analysis, qualitative analysis, experts' knowledge and investors' subjective opinions can be better integrated into a portfolio selection model. Ramaswamy (1998), Inuiguchi and Ramik (2000), Vercher *et al.* (2007), Li *et al.* (2010), Bhattacharyya *et al.* (2009, 2011), Bhattacharyya and Kar (2011) and others have made significant contributions to model fuzzy portfolio selection problem.

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In possibilistic portfolio selection models two type of approaches are noticed. The return of a security is considered either as a possibilistic variable or as a fuzzy number. In the later case, the possibilistic moments of the fuzzy numbers are considered. Possibilistic portfolio models integrate the past security data and experts' judgment to catch variations of stock markets more plausibly. Tanaka and Guo (1999), Tanaka *et al.* (2000) propose two kinds of portfolio selection models by utilizing fuzzy probabilities and exponential possibility distributions, respectively. Inuiguchi and Tanino (2000) introduce a possibilistic programming approach to the portfolio selection problem under the minimax regret criterion. Ida (2004) investigates portfolio selection problem with interval and fuzzy coefficients, two kinds of efficient solutions are introduced: possibly efficient solution as an optimistic solution, necessity efficient solution as a pessimistic solution. Carlsson and Fuller (2001) introduce a possibilistic approach for selecting portfolios with the highest utility value under the assumption that the returns of assets are trapezoidal fuzzy numbers. Wang *et al.* (2005) and Zhang and Wang (2005) discuss the general weighted possibilistic portfolio selection problems. Zhang *et al.* (2007) assume that the rates of return of assets can be expressed by possibility distribution. They propose two types of portfolio selection models based on upper and lower possibilistic means and possibilistic variances and introduce the notions of lower and upper possibilistic efficient portfolios. Li and Xu (2007) deal with a possibilistic portfolio selection problem with interval center values. By using modality approach and goal attainment approach they convert it into a nonlinear goal programming problem.

Though a considerable number of research papers have been published for portfolio selection problem in fuzzy environment, no one has considered weighted possibilistic mean- variance- skewness model for fuzzy portfolio selection. In this paper, the returns of security are considered as fuzzy numbers. The concept of weighted possibilistic moments of fuzzy numbers [Saeidifar and Pasha (2009)] are used to obtain the possibilistic mean, variance and skewness of fuzzy numbers in section 2. In section 3, these results are employed to constitute the weighted possibilistic mean- variance- skewness models for portfolio selection problem. Four different models are constructed for fuzzy portfolio optimization. The models have the three estimators: mean, variance and skewness. None of the estimators are new. But the approach to find out the values of the estimators by weighted possibilistic moments is a novice one. As discussed earlier, attempts to use different approaches to find possibilistic moments have been noticed in the literatures of portfolio selection problem. Points to note are:

1. in these attempts, none has considered skewness.
2. no attempt with weighted possibilistic moments has been noticed.

In section 4, in order to solve the models under fuzzy environment, fuzzy simulation (FS) and elitist genetic algorithm (EGA) are integrated to produce more powerful and effective hybrid intelligent algorithm (HIA). In section 5, share price data from Bombay Stock Exchange (BSE), India are used to illustrate the effectiveness of the algorithm and finally in section 6 some conclusions are specified.

2. POSSIBILISTIC MOMENTS OF FUZZY NUMBERS

In this section we will first discuss some basic concepts and theorems on possibilistic moments of fuzzy numbers [*cf.* Thavaneswaran *et al.* (2009), Saeidifar and Pasha (2009), Carlson *et al.* (2001), Fullér *et al.* (2003)]. These are crucial for the construction of this paper. After that, some theorems are developed.

Definition 2.1 A fuzzy number $\tilde{A}$ in parametric form is a pair $[\underline{a}(\alpha), \overline{a}(\alpha)]$ of functions $\underline{a}(\alpha)$ and $\overline{a}(\alpha)$, $0 \le a \le 1$ which satisfies the following requirements.

a) $\underline{a}(\alpha)$ is a bounded increasing left continuous function;

b) $\overline{a}(\alpha)$ is a bounded decreasing right continuous function;

c) $\underline{a}(\alpha) \le \overline{a}(\alpha),\ 0 \le \alpha \le 1.$

A popular fuzzy number $\tilde{A} = (a, b, c, d)$ [called trapezoidal fuzzy number] with membership function $\mu_{\tilde{A}}(x)$ is defined as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a}{b-a} & x \in [a, b] \\ 1 & x \in [b, c] \\ \dfrac{d-x}{d-c} & x \in [c, d] \\ 0 & \text{otherwise.} \end{cases}$$

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Its *a*-level sets are $[\tilde{A}]_\alpha = [\underline{a}(\alpha), \overline{a}(\alpha)] = [a + (b - a)\alpha, d - (d - c)\alpha]$.

Note that if $b = c$, then $\mu_{\tilde{A}}(x)$ is membership function of triangular fuzzy number $\tilde{A} = (a, b, d)$.

Definition 2.2 Let Ω be a nonempty set and $P(\Omega)$ is the power set of Ω. A function *POS* is called a possibility measure if

i) $POS\{\Omega\} = 1$,

ii) $POS\{\Phi\} = 0$,

iii) $POS\{\cup_i A_i\} = \sup_i POS\{A_i\}$, for any collection $\{A_i\}$ in $P(\Omega)$.

The triplet (Ω, $P(\Omega)$, *POS*) is called a possibility space.

Definition 2.3 A fuzzy variable $\tilde{r}$ is defined as a function from a possibility space (Ω, $P(\Omega)$, *POS*) to the set of real numbers $\Re$.

Definition 2.4 Let $[\tilde{A}]_\alpha = [\underline{a}(\alpha), \overline{a}(\alpha)]$ be the *a* cut of a fuzzy number $\tilde{A}$ and *f(a)* be a weighted function. Also let $C_L \le C_U \in \Re$. Then the n^{th} weighted double possibilistic moments of fuzzy number $\tilde{A}$ about points C_L, C_U are defined as:

$$M_n^{(C_L, C_U)}(\tilde{A}) = \frac{1}{2}\int_0^1 f(\alpha)[(\underline{a}(\alpha) - C_L)^n + (\overline{a}(\alpha) - C_U)^n]d\alpha,\ n = 1, 2, 3, ...$$

Double moments explain the variation of a fuzzy number with respect to points C_L and C_U. These moments are independent with respect to the points C_L, C_U. So the n^{th} weighted double possibilistic moments of fuzzy number $\tilde{A}$ about points $m_f^-(\tilde{A}), m_f^+(\tilde{A})$ from the nearest weighted interval $NWPI_f(\tilde{A}) = [m_f^-(\tilde{A}), m_f^+(\tilde{A})]$ are defined as:

$$M_n^{(m_f^-(\tilde{A}), m_f^+(\tilde{A}))}(\tilde{A}) = \frac{1}{2}\int_0^1 f(\alpha)[(\underline{a}(\alpha) - m_f^-(\tilde{A}))^n + (\overline{a}(\alpha) - m_f^+(\tilde{A}))^n]d\alpha,\ n = 1, 2, 3, ...$$

$M_n^{(m_f^-(\tilde{A}), m_f^+(\tilde{A}))}(\tilde{A})$ explains the variation of a fuzzy number $\tilde{A}$ with respect to two important points $m_f^-(\tilde{A}), m_f^+(\tilde{A})$ from support function.

If $m_f^-(\tilde{A}) = m_f^+(\tilde{A}) = \overline{m}_f(\tilde{A})$, then $M_n(\tilde{A}) = M_n^{\overline{m}_f(\tilde{A})}(\tilde{A})$ is called the n^{th} weighted possibilistic moment about the possibilistic mean value of fuzzy number $\tilde{A}$.

If $f(a) = 2a$, then $M_n(\tilde{A})$ are called the possibilistic moments about the possibilistic mean value of fuzzy number $\tilde{A}$ and it is as:

$$M_n(\tilde{A}) = \int_0^1 \alpha[(\underline{a}(\alpha) - m(\tilde{A}))^n + (\overline{a}(\alpha) - m(\tilde{A}))^n]d\alpha;\ \ n = 1, 2, 3, ...$$

where $m(\tilde{A})$ is called weighted possibilistic mean value of fuzzy number $\tilde{A}$ so that

$$m(\tilde{A}) = \int_0^1 \alpha[(\underline{a}(\alpha) + \overline{a}(\alpha)]d\alpha.$$

Definition 2.5 The second possibilistic moment $M_2(\tilde{A})$ is called the possibilistic variance of fuzzy number $\tilde{A}$ and it is defined as:

$$M_2(\tilde{A}) = \sigma^2 = Var(\tilde{A}) = \int_0^1 \alpha[(\underline{a}(\alpha) - m(\tilde{A}))^2 + (\overline{a}(\alpha) - m(\tilde{A}))^2]d\alpha\ .$$

Definition 2.6 Let $M_3(\tilde{A})$ be the possibilistic third order moment about the possibilistic mean. Then the weighted possibilistic skewness (*WPS*) of fuzzy number $\tilde{A}$ is defined as follows:

$$WPS(\tilde{A}) = \frac{M_3(\tilde{A})}{(\sqrt{M_2(\tilde{A})})^3}.$$

The *WPS* of a fuzzy number shows the weight of fuzzy number at the left or right sides of the mean value.

Note that a fuzzy number is said to be symmetric if it can be folded along an axis so that the two sides coincide with each other. A fuzzy number that lacks the symmetry with respect to a vertical axis is to be skewed.

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Theorem 2.7 Let $\tilde{A} = (a, b, c, d)$ be a trapezoidal fuzzy number. Then the possibilistic mean, variance and skewness of $\tilde{A}$ are respectively given by

$$WPM = \frac{1}{6}[\ a + 2(b + c) + d],$$

$$WPV = \frac{1}{36}[\ 2(a^2 + d^2) + 5(b^2 + c^2) + 2(ab + cd - da) - 4(ac + bd) - 8bc],$$

$$WPS = \frac{1}{5}[2(a^2 + d^2) + 5(b^2 + c^2) + 2(ab + cd - da) - 4(ac + bd) - 8bc]^{-3/2}[19(a^3 + d^3)$$
$$+26(b^3 + c^3) - 15(a^2d + ad^2) - 30(b^2c + bc^2) + 60(a + d)bc + 30(b + c)ad$$
$$-12(a^2b + cd^2) - 30(a^2c + bd^2) - 33(ab^2 + c^2d) - 15(a^2c + b^2d)].$$

Proof: Let $\tilde{A} = (a, b, c, d)$ be a trapezoidal fuzzy number.

Its α-level sets are $[\tilde{A}]_\alpha = [\underline{a}(\alpha), \overline{a}(\alpha)] = [\ a + (b - a)\alpha, d - (d - c)\alpha\ ].$

The weighted possibilistic mean value of the fuzzy number $\tilde{A}$ is

$$WPM = m(\tilde{A}) = \int_0^1 \alpha[(\ \underline{a}(\alpha) + \overline{a}(\alpha)]d\alpha = \int_0^1 \alpha[\{\ a + (b - a)\alpha\} + \{d - (d - c)\alpha\}]d\alpha$$
$$= \int_0^1 [\ (a+d)\alpha + (b + c - a - d)\alpha^2\ \}]d\alpha = \frac{1}{6}[\ a + 2(b + c) + d].$$

The weighted possibilistic variance of the fuzzy number $\tilde{A}$ is

$$WPV = M_2(\tilde{A}) = \int_0^1 \alpha[(\ \underline{a}(\alpha) - m(\tilde{A}))^2 + (\ \overline{a}(\alpha) - m(\tilde{A}))^2]d\alpha$$
$$= \int_0^1 \alpha[(\ [a + (b - a)\alpha] - (\frac{1}{6}[\ a + 2(b + c) + d]))^2 + (\ [d - (d - c)\alpha] - \frac{1}{6}[\ a + 2(b + c) + d]))^2]d\alpha$$
$$= \frac{1}{36}[\ 2(a^2 + d^2) + 5(b^2 + c^2) + 2(ab + cd - da) - 4(ac + bd) - 8bc].$$

The third possibilistic moment of the fuzzy number $\tilde{A}$ is

$$M_3(\tilde{A}) = \int_0^1 \alpha[(\ \underline{a}(\alpha) - m(\tilde{A}))^3 + (\ \overline{a}(\alpha) - m(\tilde{A}))^3]d\alpha$$
$$= \int_0^1 \alpha[(\ [a + (b - a)\alpha] - (\frac{1}{6}[\ a + 2(b + c) + d]))^3 + (\ [d - (d - c)\alpha] - \frac{1}{6}[\ a + 2(b + c) + d]))^3]d\alpha$$
$$= \frac{1}{1080}[\ 19(a^3 + d^3) + 26(b^3 + c^3) - 15(a^2d + ad^2) - 30(b^2c + bc^2) + 60(a + d)bc + 30(b + c)ad$$
$$-12(a^2b + cd^2) - 30(a^2c + bd^2) - 33(ab^2 + c^2d) - 15(a^2c + b^2d)].$$

Therefore the weighted possibilistic skewness of the trapezoidal fuzzy number $\tilde{A}$ is obtained by

$$WPS(\tilde{A}) = \frac{M_3(\tilde{A})}{(\sqrt{M_2(\tilde{A})})^3}.$$

Corollary 2.8 Let $\tilde{A} = (a, b, c)$ be a triangular fuzzy number. Then the weighted possibilistic mean, variance and skewness of $\tilde{A}$ are respectively given by

$$WPM = \frac{1}{6}[\ a + 4b + c],$$

$$WPV = \frac{1}{18}[\ a^2 + b^2 + c^2 - ab - bc - ca\ ],$$

$$WPS = \frac{19(a^3 + c^3) - 8b^3 - 42b(a^2 + c^2) + 12b^2(a + c) - 15(a^2c + ac^2) + 60abc}{10\sqrt{2}(\sqrt{[\ a^2 + b^2 + c^2 - ab - bc - ca\ ]}\,)^3}.$$

Proof: It is obvious from theorem 2.7.

Theorem 2.9 Let $\tilde{A}$ be a fuzzy number having the membership function $\mu_{\tilde{A}}(x) = exp[-(x_i - \theta)^2 / 2\sigma^2\ ],$ where θ and σ are parameters to be specified. Then the possibilistic mean, variance and skewness of $\tilde{A}$ are respectively $\theta,\ \sigma^2$ and 0.

Proof: As $\tilde{A}$ is a fuzzy number having the membership function $\mu_{\tilde{A}}(x) = exp[-(x_i - \theta)^2 / 2\sigma^2\ ],$ its α-level sets are $[\tilde{A}]_\alpha = [\underline{a}(\alpha), \overline{a}(\alpha)] = [\ \theta - \sigma\sqrt{-2.ln(-\alpha)}, \theta + \sigma\sqrt{-2.ln(-\alpha)}\ ].$

The weighted possibilistic mean value of the fuzzy number $\tilde{A}$ is

$$WPM = m(\tilde{A}) = \int_0^1 \alpha[(\ \underline{a}(\alpha) + \overline{a}(\alpha)]d\alpha$$

$$= \int_0^1 \alpha[\{\ \theta - \sigma\sqrt{-2.ln(-\alpha)}\ \} + \{\theta + \sigma\sqrt{-2.ln(-\alpha)}\ \}]d\alpha = \int_0^1 \alpha[\ 2\theta\ ]d\alpha = \theta.$$

The weighted possibilistic variance of the fuzzy number $\tilde{A}$ is

$$WPV = M_2(\tilde{A}) = \int_0^1 \alpha[(\ \underline{a}(\alpha) - m(\tilde{A}))^2 + (\ \overline{a}(\alpha) - m(\tilde{A}))^2\ ]d\alpha$$

$$= \int_0^1 \alpha[(\ [\theta - \sigma\sqrt{-2.ln(-\alpha)}\ ] - \theta)^2 + (\ [\theta + \sigma\sqrt{-2.ln(-\alpha)}\ ] - \theta))^2\ ]d\alpha = \sigma^2.$$

The third possibilistic moment of the fuzzy number $\tilde{A}$ is

$$M_3(\tilde{A}) = \int_0^1 \alpha[(\ \underline{a}(\alpha) - m(\tilde{A}))^3 + (\ \overline{a}(\alpha) - m(\tilde{A}))^3\ ]d\alpha$$

$$= \int_0^1 \alpha[(\ [\theta - \sigma\sqrt{-2.ln(-\alpha)}\ ] - \theta)^3 + (\ [\theta + \sigma\sqrt{-2.ln(-\alpha)}\ ] - \theta))^3\ ]d\alpha = 0.$$

Therefore the weighted possibilistic skewness of the trapezoidal fuzzy number $\tilde{A}$ is obtained by

$$WPS(\tilde{A}) = \frac{M_3(\tilde{A})}{(\sqrt{M_2(\tilde{A})}\,)^3} = 0.$$

3. WEIGHTED POSSIBILISTIC MEAN – VARIANCE – SKEWNESS PORTFOLIO SELECTION MODELS

Let $\tilde{r}_i$ be a fuzzy number representing the return of the i^{th} security. Let x_i be the portion of the total capital invested in security i, $i = 1, 2, \ldots, n$. Then $\tilde{r}_i = (\ p' + d_i - p_i\ ) / p_i$, where p_i is the closing price of the i^{th} security at present, p' is the estimated closing price in the next year and d_i is the estimated dividends in the next year.

Now when minimum expected return and maximum risk are known, the investor will prefer a portfolio with large skewness. It can be modeled as:

$$\begin{cases} maximize\ S[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + ... + \tilde{r}_n x_n] \\ subject\ to \\ \qquad E[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + ... + \tilde{r}_n x_n] \ \geq\ \alpha \\ \qquad V[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + ... + \tilde{r}_n x_n] \ \leq\ \gamma \\ \qquad x_1 + x_2 + ... + x_n \ =\ 1 \\ \qquad x_i \ \geq\ 0,\ i\ =\ 1,\ 2,\ ....,\ n. \end{cases} \tag{1}$$

The first constraint ensures that the expected return is not less than a, and the second constraint ensures that the risk dose not exceed some given level γ, the investor can bear. The last two constraints assure that all the capital will be invested to n securities and short selling is not allowed.

When expected return and skewness are both not less than some given target values, the investor would aim to minimize the risk. This can be modeled as:

$$\begin{cases} minimize\ V[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + ... + \tilde{r}_n x_n] \\ subject\ to \\ \qquad E[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + ... + \tilde{r}_n x_n] \ \geq\ \alpha \\ \qquad S[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + ... + \tilde{r}_n x_n] \ \geq \beta \\ \qquad x_1 + x_2 + ... + x_n \ =\ 1 \\ \qquad x_i \ \geq\ 0,\ i\ =\ 1,\ 2,\ ....,\ n\ . \end{cases} \tag{2}$$

When minimum skewness and maximum risk are known, the investor would aim to maximize the expected return. This can be modeled as:

$$\begin{cases} maximize\ E[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + ... + \tilde{r}_n x_n] \\ subject\ to \\ \qquad V[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + ... + \tilde{r}_n x_n] \ \leq\ \gamma \\ \qquad S\ [\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + ... + \tilde{r}_n x_n] \ \geq \beta \\ \qquad x_1 + x_2 + ... + x_n \ =\ 1 \\ \qquad x_i \ \geq\ 0,\ i\ =\ 1,\ 2,\ ....,\ n. \end{cases} \tag{3}$$

Finally all the three models can be composed together to form the following multi objective non-linear programming problem as:

$$\begin{cases} maximize\ E[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + ... + \tilde{r}_n x_n] \\ minimize\ V[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + ... + \tilde{r}_n x_n] \\ maximize\ S[\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + ... + \tilde{r}_n x_n] \\ subject\ to \\ \qquad x_1 + x_2 + ... + x_n \ =\ 1 \\ \qquad x_i \ \geq\ 0,\ i\ =\ 1,\ 2,\ ....,\ n. \end{cases} \tag{4}$$

Note that when the membership functions of $\tilde{r}_i$ are all symmetric, $S\ [\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + ... + \tilde{r}_n x_n\ ] = 0$ for all $x_i \geq 0,\ i = 1, 2, 3, \ldots, n$.

Theorem 3.1 Suppose $\tilde{r}_i = (a_i , b_i , c_i , d_i\ )$ are independent trapezoidal fuzzy numbers. Then model (3) generates the deterministic programming problem model (5).

$$
\left\{
\begin{array}{l}
\text{maximize } \dfrac{1}{6}\left[\sum_{i=1}^{n} a_i x_i + 2\left(\sum_{i=1}^{n} b_i x_i + \sum_{i=1}^{n} c_i x_i\right) + \sum_{i=1}^{n} d_i x_i\right] \\
\text{subject to} \\
2\left(\left(\sum_{i=1}^{n} a_i x_i\right)^2 + \left(\sum_{i=1}^{n} d_i x_i\right)^2\right) + 5\left(\left(\sum_{i=1}^{n} b_i x_i\right)^2 + \left(\sum_{i=1}^{n} c_i x_i\right)^2\right) \\
+ 2\left(\left(\sum_{i=1}^{n} a_i x_i\right)\left(\sum_{i=1}^{n} b_i x_i\right) + \left(\sum_{i=1}^{n} c_i x_i\right)\left(\sum_{i=1}^{n} d_i x_i\right) - \left(\sum_{i=1}^{n} d_i x_i\right)\left(\sum_{i=1}^{n} a_i x_i\right)\right) \\
- 4\left(\left(\sum_{i=1}^{n} a_i x_i\right)\left(\sum_{i=1}^{n} c_i x_i\right) + \left(\sum_{i=1}^{n} b_i x_i\right)\left(\sum_{i=1}^{n} d_i x_i\right)\right) - 8\left(\sum_{i=1}^{n} b_i x_i\right)\left(\sum_{i=1}^{n} c_i x_i\right) \le 36\gamma \\
\\
19\left(\left(\sum_{i=1}^{n} a_i x_i\right)^3 + \left(\sum_{i=1}^{n} d_i x_i\right)^3\right) + 26\left(\left(\sum_{i=1}^{n} b_i x_i\right)^3 + \left(\sum_{i=1}^{n} c_i x_i\right)^3\right) - 15\left(\left(\sum_{i=1}^{n} a_i x_i\right)^2\left(\sum_{i=1}^{n} d_i x_i\right) + \left(\sum_{i=1}^{n} a_i x_i\right)\left(\sum_{i=1}^{n} d_i x_i\right)^2\right) \\
-30\left(\left(\sum_{i=1}^{n} b_i x_i\right)^2\left(\sum_{i=1}^{n} c_i x_i\right) + \left(\sum_{i=1}^{n} b_i x_i\right)\left(\sum_{i=1}^{n} c_i x_i\right)^2\right) + 60\left(\left(\sum_{i=1}^{n} a_i x_i\right) + \left(\sum_{i=1}^{n} d_i x_i\right)\right)\left(\sum_{i=1}^{n} b_i x_i\right)\left(\sum_{i=1}^{n} c_i x_i\right) \\
+30\left(\left(\left(\sum_{i=1}^{n} b_i x_i\right) + \left(\sum_{i=1}^{n} c_i x_i\right)\right)\left(\sum_{i=1}^{n} a_i x_i\right)\left(\sum_{i=1}^{n} d_i x_i\right)\right) - 12\left(\left(\sum_{i=1}^{n} a_i x_i\right)^2\left(\sum_{i=1}^{n} b_i x_i\right) + \left(\sum_{i=1}^{n} c_i x_i\right)\left(\sum_{i=1}^{n} d_i x_i\right)^2\right) \\
-30\left(\left(\sum_{i=1}^{n} a_i x_i\right)^2\left(\sum_{i=1}^{n} c_i x_i\right) + \left(\sum_{i=1}^{n} b_i x_i\right)\left(\sum_{i=1}^{n} d_i x_i\right)^2\right) - 33\left(\left(\sum_{i=1}^{n} b_i x_i\right)^2\left(\sum_{i=1}^{n} a_i x_i\right) + \left(\sum_{i=1}^{n} d_i x_i\right)\left(\sum_{i=1}^{n} c_i x_i\right)^2\right) \\
-15\left(\left(\sum_{i=1}^{n} a_i x_i\right)^2\left(\sum_{i=1}^{n} c_i x_i\right) + \left(\sum_{i=1}^{n} b_i x_i\right)\left(\sum_{i=1}^{n} d_i x_i\right)^2\right) \ge 5\beta\{2\left(\left(\sum_{i=1}^{n} a_i x_i\right)^2 + \left(\sum_{i=1}^{n} d_i x_i\right)^2\right) \\
+ 5\left(\left(\sum_{i=1}^{n} b_i x_i\right)^2 + \left(\sum_{i=1}^{n} c_i x_i\right)^2\right) + 2\left(\left(\sum_{i=1}^{n} a_i x_i\right)\left(\sum_{i=1}^{n} b_i x_i\right) + \left(\sum_{i=1}^{n} c_i x_i\right)\left(\sum_{i=1}^{n} d_i x_i\right) - \left(\sum_{i=1}^{n} d_i x_i\right)\left(\sum_{i=1}^{n} a_i x_i\right)\right) \\
- 4\left(\left(\sum_{i=1}^{n} a_i x_i\right)\left(\sum_{i=1}^{n} c_i x_i\right) + \left(\sum_{i=1}^{n} b_i x_i\right)\left(\sum_{i=1}^{n} d_i x_i\right)\right) - 8\left(\sum_{i=1}^{n} b_i x_i\right)\left(\sum_{i=1}^{n} c_i x_i\right)\}^{3/2} \\
\\
x_1 + x_2 + \ldots + x_n = 1 \\
x_i \ge 0,\ i = 1, 2, \ldots, n.
\end{array}
\right. \tag{5}
$$

Proof: Since $\tilde{r}_i = (a_i, b_i, c_i, d_i)$ are trapezoidal fuzzy numbers, by Extension Principle of Zadeh (1978) it follows that

$$\tilde{r}_1 x_1 + \tilde{r}_2 x_2 + \ldots + \tilde{r}_n x_n = \left(\sum_{i=1}^{n} a_i x_i ,\ \sum_{i=1}^{n} b_i x_i ,\ \sum_{i=1}^{n} c_i x_i ,\ \sum_{i=1}^{n} d_i x_i\right),$$

which is also a trapezoidal fuzzy number. Combining this with the results obtained in theorem 2.7, we are with the theorem.

Theorem 3.2 Suppose $\tilde{r}_i = (a_i, b_i, c_i, d_i)$ are independent trapezoidal fuzzy numbers. Then model *(3.4)* generates the multi-objective programming problem model (6).

$$
\left\{
\begin{array}{l}
\textit{maximize}\ \dfrac{1}{6}\left[\sum_{i=1}^{n} a_i x_i + 2\left(\sum_{i=1}^{n} b_i x_i + \sum_{i=1}^{n} c_i x_i\right) + \sum_{i=1}^{n} d_i x_i\right] \\
\textit{minimize}\ \dfrac{1}{36}\Big[\, 2\left(\left(\sum_{i=1}^{n} a_i x_i\right)^2 + \left(\sum_{i=1}^{n} d_i x_i\right)^2\right) + 5\left(\left(\sum_{i=1}^{n} b_i x_i\right)^2 + \left(\sum_{i=1}^{n} c_i x_i\right)^2\right) \\
\qquad + 2\left(\left(\sum_{i=1}^{n} a_i x_i\right)\left(\sum_{i=1}^{n} b_i x_i\right) + \left(\sum_{i=1}^{n} c_i x_i\right)\left(\sum_{i=1}^{n} d_i x_i\right) - \left(\sum_{i=1}^{n} d_i x_i\right)\left(\sum_{i=1}^{n} a_i x_i\right)\right) \\
\qquad - 4\left(\left(\sum_{i=1}^{n} a_i x_i\right)\left(\sum_{i=1}^{n} c_i x_i\right) + \left(\sum_{i=1}^{n} b_i x_i\right)\left(\sum_{i=1}^{n} d_i x_i\right)\right) - 8\left(\sum_{i=1}^{n} b_i x_i\right)\left(\sum_{i=1}^{n} c_i x_i\right)\Big] \\
\textit{maximize}\ \dfrac{1}{5}\Big[2\left(\left(\sum_{i=1}^{n} a_i x_i\right)^2 + \left(\sum_{i=1}^{n} d_i x_i\right)^2\right) + 5\left(\left(\sum_{i=1}^{n} b_i x_i\right)^2 + \left(\sum_{i=1}^{n} c_i x_i\right)^2\right) \\
+ 2\left(\left(\sum_{i=1}^{n} a_i x_i\right)\left(\sum_{i=1}^{n} b_i x_i\right) + \left(\sum_{i=1}^{n} c_i x_i\right)\left(\sum_{i=1}^{n} d_i x_i\right) - \left(\sum_{i=1}^{n} d_i x_i\right)\left(\sum_{i=1}^{n} a_i x_i\right)\right) \\
- 4\left(\left(\sum_{i=1}^{n} a_i x_i\right)\left(\sum_{i=1}^{n} c_i x_i\right) + \left(\sum_{i=1}^{n} b_i x_i\right)\left(\sum_{i=1}^{n} d_i x_i\right)\right) - 8\left(\sum_{i=1}^{n} b_i x_i\right)\left(\sum_{i=1}^{n} c_i x_i\right)\Big]^{(-3/2)} .\Big[\, 19\left(\left(\sum_{i=1}^{n} a_i x_i\right)^3 + \left(\sum_{i=1}^{n} d_i x_i\right)^3\right) \\
+26\left(\left(\sum_{i=1}^{n} b_i x_i\right)^3 + \left(\sum_{i=1}^{n} c_i x_i\right)^3\right) - 15\left(\left(\sum_{i=1}^{n} a_i x_i\right)^2\left(\sum_{i=1}^{n} d_i x_i\right) + \left(\sum_{i=1}^{n} a_i x_i\right)\left(\sum_{i=1}^{n} d_i x_i\right)^2\right) \\
-30\left(\left(\sum_{i=1}^{n} b_i x_i\right)^2\left(\sum_{i=1}^{n} c_i x_i\right) + \left(\sum_{i=1}^{n} b_i x_i\right)\left(\sum_{i=1}^{n} c_i x_i\right)^2\right) + 60\left(\left(\sum_{i=1}^{n} a_i x_i\right) + \left(\sum_{i=1}^{n} d_i x_i\right)\right)\left(\sum_{i=1}^{n} b_i x_i\right)\left(\sum_{i=1}^{n} c_i x_i\right) \\
+30\left(\left(\left(\sum_{i=1}^{n} b_i x_i\right) + \left(\sum_{i=1}^{n} c_i x_i\right)\right)\left(\sum_{i=1}^{n} a_i x_i\right)\left(\sum_{i=1}^{n} d_i x_i\right)\right) - 12\left(\left(\sum_{i=1}^{n} a_i x_i\right)^2\left(\sum_{i=1}^{n} b_i x_i\right) + \left(\sum_{i=1}^{n} c_i x_i\right)\left(\sum_{i=1}^{n} d_i x_i\right)^2\right) \\
-30\left(\left(\sum_{i=1}^{n} a_i x_i\right)^2\left(\sum_{i=1}^{n} c_i x_i\right) + \left(\sum_{i=1}^{n} b_i x_i\right)\left(\sum_{i=1}^{n} d_i x_i\right)^2\right) \\
-33\left(\left(\sum_{i=1}^{n} b_i x_i\right)^2\left(\sum_{i=1}^{n} a_i x_i\right) + \left(\sum_{i=1}^{n} d_i x_i\right)\left(\sum_{i=1}^{n} c_i x_i\right)^2\right) - 15\left(\left(\sum_{i=1}^{n} a_i x_i\right)^2\left(\sum_{i=1}^{n} c_i x_i\right) + \left(\sum_{i=1}^{n} b_i x_i\right)^2\left(\sum_{i=1}^{n} d_i x_i\right)\right)\Big] \\
x_1 + x_2 + ... + x_n = 1 \\
x_i \ \geq\ 0,\ \mathrm{i} = 1, 2, ..., \mathrm{n}.
\end{array}
\right. \tag{6}
$$

Note that models (1) and (2) can be constructed similarly.

4. HYBRID INTELLIGENT ALGORITHM

Genetic algorithms (GAs) are stochastic search methods based on the principles of natural genetic systems. They perform a multidimensional search in providing an optimal solution for evaluation function of an optimization problem. Since Holland (1975) first proposed it, genetic algorithm has been widely studied, experimented and applied in many fields like operations research, finance, industrial engineering, VLSI design, pattern recognition, image processing etc.

While solving an optimization problem using genetic algorithms, each solution is coded as a string of finite length over a finite alphabet. Each string is considered as individual. A collection of M (finite) such individual is called a population. Genetic algorithms start with a randomly generated population of size M. In each of the iterations a new population of the same size is generated using three basic operations on the individuals of the population. The operations are selection, crossing over and mutation. The new population obtained after selection, cross over and mutation is then used to generate another population. The number of possible population is always finite since the alphabet is a finite set and M is always finite. If the knowledge about the best string is preserved within the population, such a model is called a genetic algorithm with an elitist model (EGA). An EGA converges to the global solution with any choice of initial population [*cf.* Bhandari *et al.,* 1996]. Integration of fuzzy simulation into GA has been introduced in detail in Huang (2006).

To find the optimal portfolio, we integrate fuzzy simulation into the EGA. Here, we summarize the hybrid intelligent algorithm as follows:

1. In the GA, a solution $x = (x_1, x_2, \ldots, x_n)$ is represented by the chromosome $C = (c_1, c_2, \ldots, c_n)$, where the genes $c_1, c_2, \ldots, c_n$ are in the interval $[0, 1]$. The matching between the solution and the chromosome is through $x_i = c_i / (c_1 + c_2 + \ldots + c_n)$, $i = 1, 2, \ldots, n$, which ensures that $x_1 + x_2 + \ldots + x_n = 1$ always holds. Randomly generate a point C from the hypercube $[0, 1]^n$. Use fuzzy simulation to calculate the values of the objective function $F(x)$. Then check the feasibility of the chromosomes. The chromosome that satisfies the constraints of the model is feasible. Take the feasible chromosomes as the initial population. Generate an initial population Q of size M.
2. Calculate the objective values for all chromosomes by fuzzy simulation. Then, give the rank order of the chromosomes according to the objective values. For minimization problem, the smaller the value of $F(x)$ is, the better the chromosome is, and the smaller the ordinal number the chromosome has. For maximization problem, the greater the value of $F(x)$ is, the better the chromosome is, and the smaller the ordinal number the chromosome has.
3. Compute the values of the rank-based evaluation function of the chromosomes, and then, the fitness of each chromosome according to the rank-based-evaluation function. Find the best chromosome C_{cur} of Q. If the best strings are not unique, then call anyone of the best strings in Q as C_{cur}.
4. Construct the matting-pool of feasible chromosomes using Gen-pool (C_{cur} belongs to Q). Perform cross over and mutation operations on the chromosomes of the matting pool and obtain a population Q_{temp} of feasible chromosomes.
   Note: If an invalid chromosome occurs in any operation then do the same operation for a maximum of say 10 times, if at any step it results in a valid chromosome then go to the next step, otherwise replace the invalid chromosome by a randomly generated valid chromosome.
5. Compare the fitness value of each string C of Q_{temp} with S_{cur}. Replace the worst string of Q_{temp} with C_{cur} if the fitness value of at least one string of Q_{temp} is less than the fitness value of S_{cur}; otherwise no replacement takes place in Q_{temp}. Rename Q_{temp} as Q.
6. Repeat steps *2* to *5* a number of times.
7. The best string obtained at the last iteration is the required solution.

5. CASE STUDY: BOMBAY STOCK EXCHANGE (BSE)

In this section we apply the proposed portfolio selection models on the data set extracted from Bombay stock exchange (BSE). Bombay Stock Exchange is the oldest stock exchange in Asia with a rich heritage of over 133 years of existence. What is now popularly known as BSE was established as "The Native Share & Stock Brokers' Association" in 1875. It is the first stock exchange in India which obtained permanent recognition (in 1956) from the Government of India under the Securities Contracts (Regulation) Act (SCRA) 1956. With demutualization, the stock exchange has two of world's prominent exchanges, Deutsche Borse and Singapore Exchange, as its strategic partners. Today, BSE is the world's number 1 exchange in terms of the number of listed companies and the world's 5th in handling of transactions through its electronic trading system. The companies listed on BSE command a total market capitalization of USD Trillion 1.06 as of July, 2009.

The BSE Index, SENSEX, is India's first and most popular stock market benchmark index. Sensex is tracked worldwide. It constitutes 30 stocks representing 12 major sectors. It is constructed on a 'free-float' methodology, and is sensitive to market movements and market realities. Apart from the SENSEX, BSE offers 23 indices, including 13 sectoral indices.

We have taken monthly share price data for sixty months (March 2003- February 2008) of just five companies which are included in Bombay Stock Exchange (BSE) index. Though any finite number of stocks can be considered, we have taken only five stocks to reduce the complexity of representation.

The Table 5.1 shows the companies name along with their return in the form of trapezoidal fuzzy numbers.

Table 5.1. Fuzzy return of stocks under BSE

Company	*Variables*	*Return*
Reliance energy	R	*(-0.008, 0.0223, 0.0501,0.0673)*
L&T	*L*	*(-0.0031,0.0287, 0.0611, 0.0866)*
Bhel	*B*	*(-0.0020,0.0282, 0.0581, 0.0832)*
Tata steel	*T*	*(0.0086, 0.0296, 0.0410, 0.0525)*
SBI	*S*	*(-0.100, 0.0217, 0.0576, 0.0789)*

Example 5.1 Considering the minimum expected return and the bearable maximum risk to be as 0.04323 and 0.007 respectively, we judge the following model:

$$\begin{cases} maximize\ S[\tilde{r}_1x_1+\tilde{r}_2x_2+\tilde{r}_3x_3+\tilde{r}_4x_4+\tilde{r}_5x_5] \\ \quad subject\ to \\ \quad E[\tilde{r}_1x_1+\tilde{r}_2x_2+\tilde{r}_3x_3+\tilde{r}_4x_4+\tilde{r}_5x_5] \geq 0.043 \\ \quad V[\tilde{r}_1x_1+\tilde{r}_2x_2+\tilde{r}_3x_3+\tilde{r}_4x_4+\tilde{r}_5x_5] \leq 0.007 \\ \quad x_1+x_2+x_3+x_4+x_5=1 \\ \quad x_i \geq 0, i=1,2,3,4,5. \end{cases}$$

Example 5.2 Considering the minimum skewness and the bearable maximum risk to be as – 0.05 and 0.007 respectively, we judge the following model:

$$\begin{cases} maximize\ E[\tilde{r}_1x_1+\tilde{r}_2x_2+\tilde{r}_3x_3+\tilde{r}_4x_4+\tilde{r}_5x_5] \\ \quad subject\ to \\ \quad V[\tilde{r}_1x_1+\tilde{r}_2x_2+\tilde{r}_3x_3+\tilde{r}_4x_4+\tilde{r}_5x_5] \leq 0.007 \\ \quad S[\tilde{r}_1x_1+\tilde{r}_2x_2+\tilde{r}_3x_3+\tilde{r}_4x_4+\tilde{r}_5x_5] \geq -0.05 \\ \quad x_1+x_2+x_3+x_4+x_5=1 \\ \quad x_i \geq 0, i=1,2,3,4,5. \end{cases}$$

Example 5.3 Considering the minimum skewness and the minimum expected return to be as – 0.05 and 0.043, we judge the following model:

$$\begin{cases} minimize\ V[\tilde{r}_1x_1+\tilde{r}_2x_2+\tilde{r}_3x_3+\tilde{r}_4x_4+\tilde{r}_5x_5] \\ \quad subject\ to \\ \quad E[\tilde{r}_1x_1+\tilde{r}_2x_2+\tilde{r}_3x_3+\tilde{r}_4x_4+\tilde{r}_5x_5] \geq 0.043 \\ \quad S[\tilde{r}_1x_1+\tilde{r}_2x_2+\tilde{r}_3x_3+\tilde{r}_4x_4+\tilde{r}_5x_5] \geq -0.05 \\ \quad x_1+x_2+x_3+x_4+x_5=1 \\ \quad x_i \geq 0, i=1,2,3,4,5. \end{cases}$$

Example 5.4 Consider the following multi objective portfolio problem:

$$\begin{cases} maximize\ E[\tilde{r}_1x_1+\tilde{r}_2x_2+\tilde{r}_3x_3+\tilde{r}_4x_4+\tilde{r}_5x_5] \\ minimize\ V[\tilde{r}_1x_1+\tilde{r}_2x_2+\tilde{r}_3x_3+\tilde{r}_4x_4+\tilde{r}_5x_5] \\ maximize\ S[\tilde{r}_1x_1+\tilde{r}_2x_2+\tilde{r}_3x_3+\tilde{r}_4x_4+\tilde{r}_5x_5] \\ subject\ to \\ \quad x_1+x_2+x_3+x_4+x_5=1 \\ \quad x_i \geq 0, i=1,2,3,4,5. \end{cases}$$

We apply the hybrid intelligent algorithm to solve the examples. The binary coding, uniform crossover, uniform mutation and the following parameters of genetic algorithms are considered: population size = *200*, crossover probability = *0.8*, mutation probability = *0.1,* chromosome length = *8n* (*8* bits for each gene), *n = 5* and population generations = *25*. The solutions of the above four examples are shown in table 5.2.

Table 5.2 Solutions of examples 5.1, 5.2, 5.3 and 5.4

Output	*Solutions of*			
	Example 5.1	*Example 5.2*	*Example 5.3*	*Example 5.4*
R (x_1)	*0*	*0*	*0*	*0*
L (x_2)	*0.4516*	*0.5973*	*0.5341*	*0.5374*
B (x_3)	*0.5484*	*0.4027*	*0.4510*	*0.4626*
T (x_4)	*0*	*0*	*0.0149*	*0*
S (x_5)	*0*	*0*	*0*	*0*
Mean	×	*0.0432*	×	*0.043133*
Variance	×	×	*0.0066*	*0.006705*
Skewness	*-0.0489*	×	×	*– 0.04953*

From table 5.2 it is clear that if the investor chooses the model in example 5.1, the person has to invest *45.16%* of the total asset to the *2nd* stock and *54.84%* of the total money to the *3rd* stock. In this scenario, the expected skewness will be – *0.0489*.

Similar explanation can be given for examples 5.2 and 5.3.

If the investor chooses the multi-objective model given in example 5.4, *53.74%* and *46.26%* of the total assets should be invested in the *2nd* and *3rd* stocks. In this case, the expected return, risk and skewness will be *0.043133*, *0.006705* and – *0.04953* respectively.

The portfolios for examples 5.1, 5.2, 5.3 and 5.4 are shown in figure 5.1.

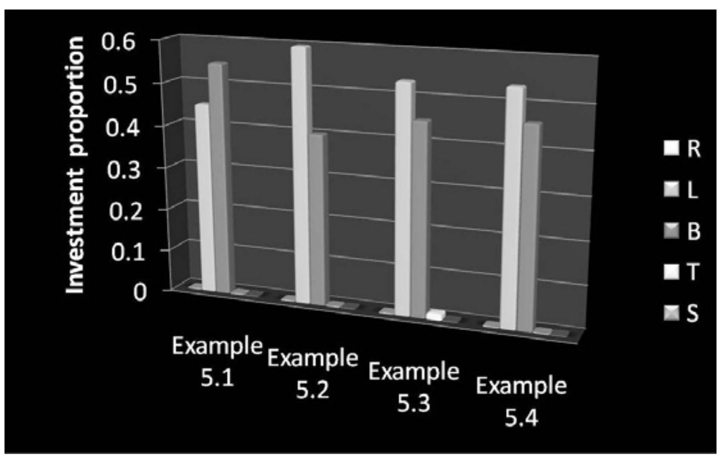

Figure 5.1 Solutions of examples 5.1, 5.2, 5.3 and 5.4

6. CONCLUSION

In this paper, the concepts of weighted possibilistic moments of fuzzy numbers are used to model the fuzzy portfolio selection problem. Four different models for fuzzy portfolio selection have been proposed considering (a) single objective optimization models, (b) tri-objective optimization model. Each problem is equivalent to a crisp parametric non linear programming problem. Integration of fuzzy simulation with elitist model of genetic algorithm is done to find a better optimal solution. Finally, we illustrated our methodology on Bombay Stock Exchange (BSE) market. In comparison to some other mean- variance- skewness portfolio selection models, this method is much less complex as this approach does not require calculating the variance- covariance and the product co-moment matrices separately form historical/statistical data. In near future, some other algorithms such as PSO (particle swarm optimization), VEGA (vector evaluation genetic algorithm), NEGA (Nondominated sorting genetic algorithm), NPGA (Niched Pareto genetic algorithm) and ACO (Ant Colony Optimization) may be employed to solve the problem, especially when the data set is significantly large.

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