International Journal of Operations Research

International Journal of Operations Research Vol. 9, No. 1, 27−43 (2012)

Optimum Cost Analysis of a Bulk Queueing System with Multiple Vacations and Restricted Admissibility of Arriving Batches

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Received November 2011; *Revised March 2012*; *Accepted March 2012*

Abstract — In this paper, a bulk queueing system with multiple vacations under a restricted admissibility policy of arriving batches is considered. Arrivals occur in bulk according to Poisson process. But all the arrivals are not considered for service. During the busy period of the server, the arrivals are admitted with probability 'α', whereas, with probability 'β', they are admitted when the server is idle. Such assumption is quite meaningful in many real life situations. The service is done in bulk with minimum of '*a*' customers and maximum of '*b*' customers. The server is assigned for secondary jobs (vacations) repeatedly when the number of waiting jobs is inadequate to process. For the proposed queueing system, the probability generating function of the steady state queue size distribution at an arbitrary time is obtained. Various performance measures are derived. A cost model for the queueing system is developed. To optimize the cost, a numerical illustration is provided.

Keywords — Bulk queue; Multiple Vacations; Restricted admissibility policy, Queue size

1. INTRODUCTION

The motivation of the model comes from a real life situation observed in an industry involving Electroplating Process (EP). Electroplating is a process that is widely used in the automotive, aerospace, electronics, medical sciences and general engineering industries. In these industries, electroplating is used for corrosion prevention, aesthetic finishes and to apply wear coatings to various components, etc. Some of the electroplating processes are hard chrome plating, nickel plating, copper plating, brass plating, etc. Detailed analysis of electro plating process can be seen in the studies of Dos Santos *et al.* (1997), Qin *et al.* (2004), Zhang *et al.* (2001, 2005), Yang (2006), Jiang (2007), Bhandari and Ma (2009), Lee (2009), etc.

The electro plating process (Hard chrome) on the components is done in bulk. Once the process is started, the bulk operation has to continue successively for many batches of metals, otherwise, the operating cost will increase. Hence, the operator will start the electroplating process only when required numbers of pieces have been accumulated for processing. After completing an electro plating process, if the number of pieces to be processed is less than the batch quantity, say '*a*', then, the operator stops the process and performs the associated works, such as rinsing, unjigging the components, buffing, inspection, etc., Further, in order to meet the customer satisfaction and to deliver the processed electroplates in time, the management may reject new order (arrivals) with some probability. The operator accepts only 'α' percent of arriving batch when the server is busy, and 'β' percent of arriving batch when the server is on vacation. This can be modeled as $M^X / G(a,b) / 1$ *queueing system with multiple vacations under a restricted admissibility policy.*

In earlier literature, on different control models of queueing systems namely, control of servers, control of service rates, control of admission of customers and control of queue discipline, one can refer Crabill, Gross and Magazine (1977), Rue and Rosenshine (1981), Stidham (1985), Neuts (1984) and Huang and Mc-Donald (1988) respectively. Madan and Abu Dayyeah (2002a, 2002b) studies some aspects of batch arrivals Bernoulli vacation models with restricted admissibility, where all arriving batches are not allowed into the system at all-time followed by Madan and Choudhury (2004a, 2004b).

Borthakur and Medhi (1974) have studied a queueing system with arrival and services in batches of variable size. They have derived the queue length distribution for the $M^X / G(a,b) / 1$ model. Krishna Reddy *et al.* (1998) have discussed $M^X / G(a,b) / 1$ model with N-policy, multiple vacations, and setup times. Arumuganathan and Jeyakumar (2005) analyzed a bulk queue with multiple vacations, setup times with N – policy and closedown times.

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Queueing systems with server vacations have attracted numerous researchers since Levy and Yechiali (1975). One of excellent survey of queueing systems with server vacations can be referred to Doshi (1986) and Takagi (1991), which includes some applications. Detailed analysis of some bulk queueing models can be seen in the studies of Chaudhry and Templeton (1983) and Medhi (1984, 2002). A batch arrival $M^X / G / 1$ queueing system with multiple vacations were first studied by Baba (1986).

Alnowibet and Tadj (2007) analyzed an *M(RA)/G(r; R)(VS)/1(BS)* queueing system such that, customers arrive at a service facility according to an orderly Poisson process, but not all arriving customers are allowed to join the system (RA: restricted admission). According to the bi-level control policy assumed, an idle period begins when the queue drops below level *r* (quorum size) and a busy period starts as soon as the queue accumulates the same number *r*. However, after each service completion, the server takes a vacation with probability *p* and starts a new service (if *r* customers are present) with probability (1 - *p*). The decisions about taking a vacation after each service completion or vacation completion are independent. The authors considered single arrival and bulk service.

Badamchi Zadeh (2009) discussed a batch arrival queue with optional second service and restricted admissibility, in which they considered a queueing system such that customers arrive at the system one by one in a compound Poisson process but not all arriving batches are allowed to join the system (restricted admission). Server provides two phases of heterogeneous service in succession. The first phase of service is essential for all customers, but as soon as the essential service is completed; a tagged customer leaves the system with probability $1 - \gamma$ $(0 \leq \gamma \leq 1)$, or moves for second phase with probability γ. The second phase has two cases (alternative) where server chooses first and second case with probability of p_1 and p_2 respectively such that $p_1 + p_2 = 1$. As soon as the first phase of a customer complete or the second phase complete, the server may go for a vacation of random length *V* with probability θ $(0 \leq \theta \leq 1)$ or it may continue to serve the next customer, if any, with probability $1 - \theta$, otherwise it remains in the system and waits for a new arrival. The authors considered bulk arrival and single service only.

Madan (2010) analyzed a batch arrival queue, with two stages of heterogeneous service, restricted admissibility of arriving batches and modified Bernoulli single vacation policy. In which, arrivals occur according to a compound Poisson process, but not all arriving batches are allowed to join the system (restricted admission). Server provides two phases of heterogeneous service with each customer having the option to choose one of the two types of first stage service followed by one of the two types of second stage service. In addition, after completion of the two stages of service in succession to each customer, the server has the option to take a vacation of a random length with probability p or to continue staying in the system with probability 1-*p*. The author considered bulk arrival and single service only.

In all the aforesaid models, the authors considered either single arrival batch service with restricted admissibility of arrivals or bulk arrival single service with restricted admissibility of arrivals. Accepting all arrivals to join into the system is not realistic always. This paper is more in general of the above models with restricted admissibility policy. *Once the arrival occurs in bulk one can expect that the service can also be done in bulk. And also it is necessary to allow the server to do secondary jobs (vacation) to optimize the overall cost.*

In this paper, the analysis of a bulk queueing system with multiple vacations under a restricted admissibility policy of arriving batches is considered. Arrivals occur in bulk according to Poisson process. But all the arrivals are not considered for service. During the busy period of the server, the arrivals are admitted with probability 'α', whereas with probability 'β', they are admitted when the server is idle. Such assumption is quite meaningful in many real life situations. The service is done in bulk with minimum of '*a*' customers and maximum of '*b*' customers. The server is assigned for secondary jobs (vacations) repeatedly when the number of waiting jobs is inadequate to the process. For the proposed queueing system, the probability generating function of the steady state queue size distribution at an arbitrary time is obtained.Various performance measures are derived. A cost model for the queueing system is developed. To optimize the cost, a numerical illustration is provided.

The following points are addressed in this paper. Control of admission of customers is considered in a bulk queueing vacation model. The model is developed in a most general way so that many existing models become particular case of the proposed model. Probability generating function (PGF) of the steady state queue size distribution at an arbitrary time epoch is obtained. Some special cases are also discussed. A cost model has been developed; and an important contribution of this is, the study of cost model for a practical situation and to optimize the cost. Various performance measures are also derived.

2. MATHEMATICAL MODEL

In this section, the steady-state equations for the system by treating the remaining service time and remaining vacation time as supplementary variables are developed.

Let X be the group size random variable of the arrival, λ be the Poisson arrival rate. g_k be the probability that '*k*' customers arrive in a batch and $X(z)$ be its probability generating function (PGF). An arriving batch is allowed to join the queue during the busy period with probability 'α'and with probability 'β' during a vacation period. Let $S(x)$ ($s(x)$) {$\tilde{S}(\theta)$}

$[S^0(x)]$ be the cumulative distribution function (probability density function) { Laplace-Stieltjes transform}[remaining service time] of service. Let $V(x)$ $(v(x))$ $\{\tilde{V}(\theta)\}$[$V^0(x)$] be the cumulative distribution function (probability density function) {Laplace - Stieltjes transform}[remaining vacation time] of vacation. $N_q(t)$ denotes the number of customers waiting for service at time t, $N_s(t)$ denotes the number of customers under the service at time t.

$$C(t)=\begin{cases}0, & \text{when the server is on vacation}\\ 1, & \text{when the server is busy with service}\end{cases}$$

$$Y(t)=j \text{ if the server is on } j^{\text{th}} \text{ vacation starting from the idle period}$$

$$P_{ij}(x,t)\,dt = \Pr\left\{N_s(t)=i, N_q(t)=j,\ x\le S^0(t)\le x+dt, C(t)=1\right\},\ \ a\le i\le b,\ j\ge 0$$

$$Q_{jn}(x,t)\,dt = \Pr\left\{N_q(t)=n,\ x\le V^0(t)\le x+dt, C(t)=0, Z(t)=j\right\},\ \ j\ge 1,\ n\ge 0$$

Now, the following system equations are obtained for the queueing system, using supplementary variable technique:

$$P_{i,0}(x-\Delta t,t+\Delta t)=P_{i,0}(x,t)\left(1-\lambda\Delta t\right)+\lambda(1-\alpha)P_{i,0}(x,t)\Delta t + \sum_{m=a}^{b}P_{m,i}(0,t)\ s(x)\Delta t+\sum_{l=1}^{\infty}Q_{l,i}(0,t)\ s(x)\Delta t; \quad a\le i\le b$$

$$P_{i,j}(x-\Delta t,t+\Delta t)=P_{i,j}(x,t)\left(1-\lambda\Delta t\right)+\lambda(1-\alpha)P_{i,j}(x,t)\Delta t+\alpha\sum_{k=1}^{j}P_{i,j-k}(x,t)\lambda g_k\Delta t;\ a\le i\le b-1\ \ \&\ \ j\ge 1$$

$$P_{b,j}(x-\Delta t,t+\Delta t)=P_{b,j}(x,t)\left(1-\lambda\Delta t\right)+\lambda(1-\alpha)P_{b,j}(x,t)\Delta t+\alpha\sum_{k=1}^{j}P_{b,j-k}(x,t)\lambda g_k\Delta t + \sum_{m=a}^{b}P_{m,b+j}(0,t)\ s(x)\Delta t+\sum_{l=1}^{\infty}Q_{l,b+j}(0,t)\ s(x)\Delta t;\quad j\ge 1$$

$$Q_{1,0}(x-\Delta t,t+\Delta t)=Q_{1,0}(x,t)\left(1-\lambda\Delta t\right)+\lambda(1-\beta)Q_{1,0}(x,t)\Delta t+\sum_{m=a}^{b}P_{m,0}(0,t)\ v(x)\Delta t$$

$$Q_{1,n}(x-\Delta t,t+\Delta t)=Q_{1,n}(x,t)\left(1-\lambda\Delta t\right)+\lambda(1-\beta)Q_{1,n}(x,t)\Delta t + \sum_{m=a}^{b}P_{m,n}(0,t)v(x)\Delta t+\beta\sum_{k=1}^{n}Q_{1,n-k}(x,t)\lambda g_k\Delta t\ ;\quad 1\le n\le a-1$$

$$Q_{1,n}(x-\Delta t,t+\Delta t)=Q_{1,n}(x,t)\left(1-\lambda\Delta t\right)+\lambda(1-\beta)Q_{1,n}(x,t)\Delta t+\beta\sum_{k=1}^{n}Q_{1,n-k}(x,t)\lambda g_k\Delta t\ ;\ n\ge a$$

$$Q_{j,0}(x-\Delta t,t+\Delta t)=Q_{j,0}(x,t)\left(1-\lambda\Delta t\right)+\lambda(1-\beta)Q_{j,0}(x,t)\Delta t+Q_{j-1,0}(0,t)v(x)\Delta t\ ;\qquad j\ge 2$$

$$Q_{j,n}(x-\Delta t,t+\Delta t)=Q_{j,n}(x,t)\left(1-\lambda\Delta t\right)+\lambda(1-\beta)Q_{j,n}(x,t)\Delta t + \beta\sum_{k=1}^{n}Q_{j,n-k}(x,t)\lambda g_k\Delta t+Q_{j-1,n}(0,t)v(x)\Delta t;\quad j\ge 2,\quad 1\le n\le a-1$$

$$Q_{j,n}(x-\Delta t,t+\Delta t)=Q_{j,n}(x,t)\left(1-\lambda\Delta t\right)+\lambda(1-\beta)Q_{j,n}(x,t)\Delta t+\beta\sum_{k=1}^{n}Q_{j,n-k}(x,t)\lambda g_k\Delta t\ ;n\ge a,\ \ j\ge 2$$

3. STEADY STATE QUEUE SIZE DISTRIBUTION

From the above equations, the steady state queue size equations are obtained as follows:

$$P_{i,0}(x-\Delta t,t+\Delta t)-P_{i,0}(x,t)=-\lambda\Delta t\ P_{i,0}(x,t)+\lambda(1-\alpha)P_{i,0}(x,t)\Delta t + \sum_{m=a}^{b}P_{m,i}(0,t)\ s(x)\Delta t+\sum_{l=1}^{\infty}Q_{l,i}(0,t)\ s(x)\Delta t$$

Dividing both sides by Δt , and letting the limit $\Delta t\to 0$, the steady state equation is obtained as

$$-\frac{d}{dx}P_{i,0}(x)=-\lambda P_{i0}(x)+\lambda(1-\alpha)P_{i0}(x)+\sum_{m=a}^{b}P_{mi}(0)s(x)+\sum_{l=1}^{\infty}Q_{li}(0)s(x)\ ;\quad a\le i\le b \tag{1}$$

Similarly, the remaining steady state equations are obtained as

$$-\frac{d}{dx}P_{i,j}(x) = -\lambda P_{ij}(x) + \lambda(1-\alpha)P_{ij}(x) + \lambda\alpha\sum_{k=1}^{j}P_{i,j-k}(x)\,g_k;\quad a \le i \le b-1\ \ \&\ j \ge 1 \tag{2}$$

$$-\frac{d}{dx}P_{b,j}(x) = -\lambda P_{bj}(x) + \lambda(1-\alpha)P_{bj}(x) + \lambda\alpha\sum_{k=1}^{j}P_{b,j-k}(x)\,g_k$$
$$+\sum_{m=a}^{b}P_{m,b+j}(0)\,s(x) + \sum_{l=1}^{\infty}Q_{l,b+j}(0)\,s(x);\quad j \ge 1 \tag{3}$$

$$-\frac{d}{dx}Q_{1,0}(x) = -\lambda Q_{10}(x) + \lambda(1-\beta)Q_{10}(x) + \sum_{m=a}^{b}P_{m0}(0)\,v(x) \tag{4}$$

$$-\frac{d}{dx}Q_{1,n}(x) = -\lambda Q_{1n}(x) + \lambda(1-\beta)Q_{1n}(x) + \sum_{m=a}^{b}P_{mn}(0)\,v(x) + \lambda\beta\sum_{k=1}^{n}Q_{1,n-k}(x)\,g_k\ ;\ 1 \le n \le a-1 \tag{5}$$

$$-\frac{d}{dx}Q_{1,n}(x) = -\lambda Q_{1n}(x) + \lambda(1-\beta)Q_{1n}(x) + \lambda\beta\sum_{k=1}^{n}Q_{1,n-k}(x)\,g_k;\quad n \ge a \tag{6}$$

$$-\frac{d}{dx}Q_{j,0}(x) = -\lambda Q_{j0}(x) + \lambda(1-\beta)Q_{j0}(x) + Q_{j-1,0}(0)\,v(x)\ ;\quad j \ge 2 \tag{7}$$

$$-\frac{d}{dx}Q_{j,n}(x) = -\lambda Q_{jn}(x) + \lambda(1-\beta)Q_{jn}(x) + \lambda\beta\sum_{k=1}^{n}Q_{j,n-k}(x)\,g_k + Q_{j-1,n}(0)\,v(x);\ j \ge 2,\ 1 \le n \le a-1 \tag{8}$$

$$-\frac{d}{dx}Q_{j,n}(x) = -\lambda Q_{jn}(x) + \lambda(1-\beta)Q_{jn}(x) + \lambda\beta\sum_{k=1}^{n}Q_{1,n-k}(x)\,g_k\ ;\ n \ge a,\ j \ge 2 \tag{9}$$

Taking Laplace-Stieltjes transform on both sides of the equation (1) through (9), we have

$$\theta\tilde{P}_{i0}(\theta) - P_{i0}(0) = \lambda\tilde{P}_{i0}(\theta) - \lambda(1-\alpha)\tilde{P}_{i0}(\theta) - \sum_{m=a}^{b}P_{mi}(0)\,\tilde{S}(\theta) - \sum_{l=1}^{\infty}Q_{li}(0)\ \tilde{S}(\theta);\quad a \le i \le b \tag{10}$$

$$\theta\tilde{P}_{ij}(\theta) - P_{ij}(0) = \lambda\tilde{P}_{ij}(\theta) - \lambda(1-\alpha)\tilde{P}_{ij}(\theta) - \lambda\alpha\sum_{k=1}^{j}\tilde{P}_{i,j-k}(\theta)\,g_k\ ;\quad a \le i \le b-1,\ j \ge 1 \tag{11}$$

$$\theta\tilde{P}_{bj}(\theta) - P_{bj}(0) = \lambda\tilde{P}_{bj}(\theta) - \lambda(1-\alpha)\tilde{P}_{bj}(\theta) - \lambda\alpha\sum_{k=1}^{j}\tilde{P}_{b,j-k}(\theta)\,g_k - \sum_{m=a}^{b}P_{m,b+j}(0)\tilde{S}(\theta)$$
$$-\sum_{l=1}^{\infty}Q_{l,b+j}(0)\ \tilde{S}(\theta)\ ;\ j \ge 1 \tag{12}$$

$$\theta\,\tilde{Q}_{10}(\theta) - Q_{10}(0) = \lambda\tilde{Q}_{10}(\theta) - \lambda(1-\beta)\tilde{Q}_{10}(\theta) - \sum_{m=a}^{b}P_{m0}(0)\tilde{V}(\theta) \tag{13}$$

$$\theta\,\tilde{Q}_{1n}(\theta) - Q_{1n}(0) = \lambda\tilde{Q}_{1n}(\theta) - \lambda(1-\beta)\tilde{Q}_{1n}(\theta) - \lambda\beta\sum_{k=1}^{n}\tilde{Q}_{1,n-k}(\theta)g_k - \sum_{m=a}^{b}P_{mn}(0)\,\tilde{V}(\theta);\ 1 \le n \le a-1 \tag{14}$$

$$\theta\,\tilde{Q}_{1n}(\theta) - Q_{1n}(0) = \lambda\tilde{Q}_{1n}(\theta) - \lambda(1-\beta)\tilde{Q}_{1n}(\theta) - \lambda\beta\sum_{k=1}^{n}\tilde{Q}_{1,n-k}(\theta)g_k;\ n \ge a \tag{15}$$

$$\theta\,\tilde{Q}_{j0}(\theta) - Q_{j0}(0) = \lambda\tilde{Q}_{j0}(\theta) - \lambda(1-\beta)\tilde{Q}_{j0}(\theta) - Q_{j-1,0}\tilde{V}(\theta)\ ;\quad j \ge 2 \tag{16}$$

$$\theta\,\tilde{Q}_{jn}(\theta) - Q_{jn}(0) = \lambda\tilde{Q}_{jn}(\theta) - \lambda(1-\beta)\tilde{Q}_{jn}(\theta) - \lambda\beta\sum_{k=1}^{n}\tilde{Q}_{j,n-k}(\theta)g_k - Q_{j-1,n}(0)\tilde{V}(\theta)\ ;\ 1 \le n \le a-1, j \ge 2 \tag{17}$$

$$\theta\,\tilde{Q}_{jn}(\theta) - Q_{jn}(0) = \lambda\tilde{Q}_{jn}(\theta) - \lambda(1-\beta)\tilde{Q}_{jn}(\theta) - \lambda\beta\sum_{k=1}^{n}\tilde{Q}_{j,n-k}(\theta)g_k\ ;\ n \ge a, j \ge 2 \tag{18}$$

To obtain the probability generating function (PGF) of the queue size at an arbitrary time, the following probability generating functions are defined.

$$\tilde{P}_i(z,\theta) = \sum_{j=1}^{\infty}\tilde{P}_{ij}(\theta)\,z^j \qquad \text{and} \qquad P_i(z,0) = \sum_{j=0}^{\infty}P_{ij}(0)\,z^j;\quad a \le i \le b$$

$$\tilde{Q}_j(z,\theta) = \sum_{n=0}^{\infty}\tilde{Q}_{jn}(\theta)\,z^n \qquad \text{and} \qquad Q_j(z,0) = \sum_{n=0}^{\infty}Q_{jn}(0)\,z^n \tag{19}$$

Using PGF and taking Z – transforms on the equations (10) – (18), we obtained the following:

$$(\theta - \beta(\lambda - \lambda X(z)))\tilde{Q}_1(z,\theta) = Q_1(z,0) - \tilde{V}(\theta)\sum_{n=0}^{a-1}\sum_{m=a}^{b} P_{mn}(0)z^n \tag{20}$$

$$(\theta - \beta(\lambda - \lambda X(z)))\tilde{Q}_j(z,\theta) = Q_j(z,0) - \tilde{V}(\theta)\sum_{n=0}^{a-1} Q_{j-1,n}(0)z^n, \quad j \geq 2 \tag{21}$$

$$(\theta - \alpha(\lambda - \lambda X(z)))\tilde{P}_i(z,\theta) = P_i(z,0) - \tilde{S}(\theta)\left[\sum_{m=a}^{b} P_{mi}(0) + \sum_{l=1}^{\infty} Q_{li}(0)\right]; \ a \leq i \leq b-1 \tag{22}$$

$$\begin{aligned} z^b(\theta - \alpha(\lambda - \lambda X(z)))\tilde{P}_b(z,\theta) = z^b P_b(z,0) &- \tilde{S}(\theta)\left\{\sum_{m=a}^{b}\left[P_m(z,0) - \sum_{j=0}^{b-1} P_{mj}(0)\, z^j\right]\right\} \\ &- \tilde{S}(\theta)\left\{\sum_{l=1}^{\infty}\left[Q_l(z,0) - \sum_{j=0}^{b-1} Q_{lj}(0)\, z^j\right]\right\} \end{aligned} \tag{23}$$

Substituting $\theta = \beta(\lambda - \lambda X(z))$ in equations (20) and (21), we get

$$Q_1(z,0) = \tilde{V}(\beta(\lambda - \lambda X(z)))\sum_{n=0}^{a-1}\sum_{m=a}^{b} P_{mn}(0)z^n \tag{24}$$

$$Q_j(z,0) = \tilde{V}(\beta(\lambda - \lambda X(z)))\sum_{n=0}^{a-1} Q_{j-1,n}(0)z^n, \quad j \geq 2 \tag{25}$$

Substituting $\theta = \alpha(\lambda - \lambda X(z))$ in equations (22) and (23), we get

$$P_i(z,0) = \tilde{S}(\alpha(\lambda - \lambda X(z)))\left[\sum_{m=a}^{b} P_{mi}(0) + \sum_{l=1}^{\infty} Q_{li}(0)\right]; \ a \leq i \leq b-1 \tag{26}$$

$$z^b P_b(z,0) = \tilde{S}(\alpha(\lambda - \lambda X(z)))\left\{\sum_{m=a}^{b}\left[P_m(z,0) - \sum_{j=1}^{b-1} P_{mj}(0)\, z^j\right] + \sum_{l=1}^{\infty}\left[Q_l(z,0) - \sum_{j=0}^{b-1} Q_{lj}(0)\, z^j\right]\right\} \tag{27}$$

Substituting $P_i(z, 0)$, $a \leq i \leq b$-1 in (27) , using (26) and then solving for $P_b(z,0)$, we get

$$P_b(z,0) = \frac{\tilde{S}\left[\alpha(\lambda - \lambda X(z))\right] f(z)}{z^b - \tilde{S}\left[\alpha(\lambda - \lambda X(z))\right]} \tag{28}$$

where

$$\begin{aligned} f(z) = \tilde{S}(\alpha(\lambda - \lambda X(z)))\sum_{i=a}^{b-1}\left[\sum_{m=a}^{b} P_{mi}(0) + \sum_{l=1}^{\infty} Q_{lj}(0)\right] - \sum_{j=0}^{b-1}\sum_{m=a}^{b} P_{mj}(0)\, z^j - \sum_{j=0}^{b-1}\sum_{l=1}^{\infty} Q_{lj}(0)\, z^j \\ + \tilde{V}(\beta(\lambda - \lambda X(z)))\left[\sum_{n=0}^{a-1}\sum_{m=a}^{b} P_{mn}(0)\, z^n + \sum_{n=0}^{a-1}\sum_{l=1}^{\infty} Q_{ln}(0)\, z^n\right] \end{aligned} \tag{29}$$

From equations (20) and (24), we have

$$\tilde{Q}_1(z,\theta) = \frac{1}{(\theta - \beta(\lambda - \lambda X(z)))}\left\{\left(\tilde{V}(\beta(\lambda - \lambda X(z))) - \tilde{V}(\theta)\right)\sum_{n=0}^{a-1}\sum_{m=a}^{b} P_{mn}(0)z^n\right\} \tag{30}$$

From equations (21) and (25), we have

$$\tilde{Q}_j(z,\theta) = \frac{1}{(\theta - \beta(\lambda - \lambda X(z)))}\left\{\left(\tilde{V}(\beta(\lambda - \lambda X(z))) - \tilde{V}(\theta)\right)\sum_{n=0}^{a-1} Q_{j-1,n}(0)z^n\right\}, \quad j \geq 2 \tag{31}$$

From equations (22) and (26), we have

$$\tilde{P}_i(z,\theta) = \frac{1}{(\theta - \alpha(\lambda - \lambda X(z)))}\left\{\left(\tilde{S}(\alpha(\lambda - \lambda X(z))) - \tilde{S}(\theta)\right)\left[\sum_{m=a}^{b} P_{mi}(0) + \sum_{l=1}^{\infty} Q_{li}(0)\right]\right\}; \ a \leq i \leq b-1 \tag{32}$$

From equations (23) and (28), we have

$$\tilde{P}_b(z,\theta) = \frac{\left[\tilde{S}(\alpha(\lambda - \lambda X(z))) - \tilde{S}(\theta)\right] f(z)}{\left[\theta - \alpha(\lambda - \lambda X(z))\right]\left[z^b - \tilde{S}(\alpha(\lambda - \lambda X(z)))\right]} \quad \text{where } f(z) \text{ is given in (29).} \tag{33}$$

Let $P(z)$ be the probability generating function of the queue size at an arbitrary time epoch. Then,

$$P(z) = \sum_{i=a}^{b-1} \tilde{P}_i(z,0) + \tilde{P}_b(z,0) + \sum_{l=1}^{\infty} \tilde{Q}_l(z,0) \text{ and let } \sum_{m=a}^{b} P_{mi}(0) = p_i \;,\; \sum_{l=1}^{\infty} Q_{li}(0) = q_i \text{ and } c_i = p_i + q_i$$

Using equations (30) – (33) in $P(z)$, we have

$$P(z) = \frac{\left\{\begin{array}{l} \beta \sum_{i=a}^{b-1} \left(\tilde{S}(\alpha(\lambda - \lambda X(z))) - 1\right)\left(z^b - z^i\right) c_i \\ + \left(\tilde{V}(\beta(\lambda - \lambda X(z))) - 1\right)\left(\beta\left(\tilde{S}(\alpha(\lambda - \lambda X(z))) - 1\right) + \alpha\left(z^b - \tilde{S}(\alpha(\lambda - \lambda X(z)))\right)\right) \sum_{n=0}^{a-1} c_n z^n \end{array}\right\}}{\alpha\beta\left(z^b - \tilde{S}(\alpha(\lambda - \lambda X(z)))\right)\left(\lambda X(z) - \lambda\right)} \tag{34}$$

The probability generating function $P(z)$ has to satisfy $P(1) = 1$. Applying L'Hospital's rule in (34), then $\rho < 1$ is the condition to be satisfied for the existence of steady state for the model under consideration, where $\rho = \dfrac{\alpha\lambda \mathrm{E}(X)\mathrm{E}(S)}{b}$.

Equation (34) gives the probability generating function $P(z)$ of the number of customers in the queue at an arbitrary time epoch, which involves 'b' unknown probabilities namely, $c_0, c_1, c_2, \ldots, c_{b-1}$. By Rouche's theorem, the expression $z^b - \tilde{S}(\alpha(\lambda - \lambda X(z)))$ has b-1 zeros inside and one on the unit circle $|z| = 1$. Since $P(z)$ is analytic within and on the unit circle, the numerator of (34) must vanish at these points, which gives 'b' equations and 'b' unknowns. These equations can be solved by suitable numerical techniques.

The unknown probabilities $q_0, q_1, q_2, \ldots, q_{a-1}$ are expressed in terms of $p_0, p_1, p_2, \ldots, p_{a-1}$ in theorem (1). which are useful to find some of the performance measures.

Theorem: 1

The constants q_n involved in $P(z)$ are expressed in terms of p_n as,

$q_n = \sum_{i=0}^{n} b_{n-i} p_i$ where $b_0 = \dfrac{\omega_0}{1 - \omega_0}$, $b_n = \dfrac{\omega_n + \sum_{j=1}^{n} b_{n-j}\,\omega_j}{1 - \beta_0}$, where ω_i is the probability that 'i' customers arrive during a vacation period.

Proof:

From the equation (19), we have $\sum_{j=1}^{\infty} Q_j(z,0) = \sum_{n=0}^{\infty} q_n z^n$

From the equations (24) and (25), we have

$$\sum_{j=1}^{\infty} Q_j(z,0) = \tilde{V}(\beta(\lambda - \lambda X(z))) \sum_{n=0}^{a-1} \left[p_n + q_n\right] z^n$$

$$\sum_{n=0}^{\infty} q_n z^n = \left(\sum_{n=0}^{\infty} \omega_n z^n\right) \sum_{n=0}^{a-1} \left(p_n + q_n\right) z^n$$

$$= \sum_{n=0}^{a-1} \left[\sum_{i=0}^{n} \omega_{n-i} \left(p_i + q_i\right)\right] z^n + \sum_{n=a}^{\infty} \left[\sum_{i=0}^{a-1} \omega_{n-i} \left(p_i + q_i\right)\right] z^n$$

Equating the coefficient of z^n, n= 0,1,2,3,…a-1, on both sides of the equation, we have $q_n = \sum_{i=0}^{n} \omega_{n-i} \left(p_i + q_i\right)$

$$q_n = \frac{\sum_{i=0}^{n} \omega_{n\text{-}i} p_i + \sum_{i=0}^{n-1} \omega_{n\text{-}i} q_i}{1 - \omega_0}$$

Coefficient of p_n in q_n is $\dfrac{\omega_0}{1-\omega_0} = b_0$

Coefficient of p_{n-1} in q_n is $[\omega_1 + \omega_1$ Coefficient of p_{n-1} in $q_{n-1}]/1-\omega_0$

$$= \frac{\omega_1 + \omega_1 b_0}{1-\omega_0} = b_1$$

Coefficient of p_{n-2} in q_n is $[\omega_2 + \omega_1$ Coefficient of p_{n-2} in q_{n-1} $+ \omega_2$ Coefficient of p_{n-2} in $q_{n-2}]/1-\omega_0$

$$= \frac{\omega_2 + (\omega_1 b_1 + \omega_2 b_0)}{1-\omega_0} = b_2$$

Proceeding like this, we get coefficient of p_0 in q_n is $\dfrac{\omega_n + \sum_{i=1}^{n} \omega_i b_{n\text{-}i}}{1-\omega_0} = b_n$

Therefore, $q_n = \sum_{i=0}^{n} b_{n-i} p_i$ (35)

4. PERFORMANCE MEASURES

In a waiting line, it is customary to access the mean number of waiting units and mean waiting time. In this section, some useful performance measures of the proposed model like expected number of customers in the queue E(Q), expected length of idle period E(I), expected length of busy period E(B) are derived which are useful to find the total average cost of the system. Also, expected waiting time in the queue W_Q, probability that the server is on vacation P(V) and probability that the server is busy P(B) are derived.

4.1 Expected Queue Length

The expected queue length E(*Q*) at an arbitrary time epoch is obtained by differentiating P(*z*) at *z* = 1 and is given by

$$\mathrm{E}(Q) = \lim_{z\to 1} P(z)$$

$$\mathrm{E}(Q) = \frac{1}{2\lambda\alpha\beta \mathrm{E}(X)\left(T1\right)^2}\left\{\begin{array}{l}\sum_{i=a}^{b-1} \beta c_i \left(b(b-1) - i(i-1)\right) f_1(X,S) + \sum_{i=a}^{b-1} \beta c_i \left(b-i\right) f_2(X,S) \\ \sum_{i=0}^{a-1} \beta c_i \left(f_3(X,S,V) - f_4(X,S,V)\right) + \sum_{i=0}^{a-1} \alpha c_i \left(f_5(X,S,V) - f_6(X,S,V)\right)\end{array}\right\} \quad (36)$$

where

$S1 = \alpha\lambda \mathrm{E}(X)\mathrm{E}(S)$; $T1 = b - S1$; $V1 = \beta\lambda \mathrm{E}(X)\mathrm{E}(V)$; $V2 = \beta\lambda X''(1)\mathrm{E}(V) + \beta^2\lambda^2 \mathrm{E}^2(X)\mathrm{E}(V^2)$

$S2 = \alpha\lambda X''(1)\mathrm{E}(S) + \alpha^2\lambda^2\mathrm{E}^2(X)\mathrm{E}(S^2)$; $S3 = \lambda X''(1)(T1) + \lambda\mathrm{E}(X)b(b-1) - \lambda\mathrm{E}(X)(S2)$;

$f_1(X,S) = (T1)(S1)$; $f_2(X,S) = (T1)(S2) - \alpha\,\mathrm{E}(S)S3$; $f_3(X,S,V) = (T1)\{2i(V1)(S1) + (S2)(V1) + (S1)(V2)\}$

$f_4(X,S,V) = (S3)(S1)\,\beta\,\mathrm{E}(V)$; $f_5(X,S,V) = (T1)\{2i(V1)(T1) + b(b-1)(V1) - (V1)(S2) + (T1)(V2)\}$

$f_6(X,S,V) = (S3)(T1)\,\beta\,\mathrm{E}(V)$;

4.2 Expected Length of Idle Period

Let *I* be the idle period random variable due to multiple vacation process. Let *U* be a random variable such that $U = 0$, if the server finds at least '*a*' customers after the first vacation and $U = 1$, if he finds less than '*a*' customers after the first vacation. Then the expected length of the idle period E(*I*) is given by

$$\begin{aligned}\mathrm{E}(I) &= \mathrm{E}(I / U = 0)\mathrm{P}(U = 0) + \mathrm{E}(I / U = 1)\mathrm{P}(U = 1) \\ &= \mathrm{E}(V)\mathrm{P}(U = 0) + (\mathrm{E}(V) + \mathrm{E}(I))\mathrm{P}(U = 1)\end{aligned}$$

and since $\mathrm{P}(U = 0) + \mathrm{P}(U = 1) = 1$, solving for E(*I*), we have $\mathrm{E}(I) = \dfrac{\mathrm{E}(V)}{\mathrm{P}(U=0)}$

From the equation (24),

$$Q_1(z,0) = \tilde{V}(\beta(\lambda - \lambda X(z)))\sum_{n=0}^{a-1} p_n z^n$$

$$\sum_{n=0}^{\infty} Q_{1n}(0)z^n = \left(\sum_{n=0}^{\infty} \omega_n z^n\right)\left(\sum_{n=0}^{a-1} p_n z^n\right)$$, where ω_i is the probability that 'i' customers arrive during a vacation.

$$\sum_{n=0}^{\infty} Q_{1n}(0)z^n = \left(\sum_{n=0}^{a-1}\sum_{i=0}^{n} \omega_i p_{n-i} z^n\right)\left(\sum_{n=a}^{\infty}\sum_{i=0}^{a-1} p_i w_{n-i} z^n\right)$$

Equating the coefficient of z^n; $n = 0,1,2,3,\ldots,a-1$, we get $Q_{1n}(0) = \sum_{i=0}^{n} \omega_i p_{n-i}$

Now, $\mathrm{P}(U = 0) = 1 - \sum_{n=0}^{a-1}\sum_{i=0}^{n} \omega_i p_{n-i}$

Thus, $$\mathrm{E}(I) = \frac{\mathrm{E}(V)}{1 - \sum_{n=0}^{a-1}\sum_{i=0}^{n} \omega_i p_{n-i}} \tag{37}$$

4.3 Expected Length of Busy Period

Let *B* be the busy period random variable. A random variable *J* is defined, as, *J*=0, if the server finds less than '*a*' customers after the first service and *J*=1, if the server finds '*a*' or more customers after the first service. Then,

E(*B*) = E(*B* / *J* = 0)P(*J* = 0) + E(*B* / *J* = 1)P(*J* = 1)
= E(*S*)P(*J* = 0) + (E(*S*) + E(*B*))P(*J* = 1)

and since P(*J* = 0) + P(*J* = 1) = 1, solving for E(*B*), we get

$$\mathrm{E}(B) = \frac{\mathrm{E}(S)}{\mathrm{P}(J=0)} = \frac{\mathrm{E}(S)}{\sum_{i=0}^{a-1} p_i} \tag{38}$$

4.4 Expected waiting time in the queue

The mean waiting time of the customers in the queue (W_Q) can be easily obtained using Little's formula

$$W_Q = \frac{\mathrm{E}(Q)}{\lambda \mathrm{E}(X)} \tag{39}$$

4.5 Probability that the server is on vacation

Let *P*(*V*) be the probability that the server is on multiple vacations at time *t*.
From the equation (30) and (31), we have

$$\sum_{j=1}^{\infty} \tilde{Q}_j(z,0) = \frac{\left(\tilde{V}(\beta(\lambda - \lambda X(z))) - 1\right)}{\left(-\lambda + \lambda X(z)\right)}\left[\sum_{n=0}^{a-1} c_n z^n\right]$$

Now, $P(V) = \lim_{z\to 1} \tilde{Q}_j(z,0)$

$$= \mathrm{E}(V)\left[\sum_{n=0}^{a-1} c_n\right] \tag{40}$$

4.6 Probability that the server is busy

Let *P*(*B*) be the probability that the server is in the busy period at time *t*.

From the equations (32) and (33), we have

$$P(B) = \lim_{z\to 1} \sum_{i=a}^{b} \tilde{P}_i(z,0)$$

$$= \lim_{z\to 1}\left[\sum_{i=a}^{b-1} \tilde{P}_i(z,0) + \tilde{P}_b(z,0)\right]$$

$$P(B) = \mathrm{E}(S)\sum_{i=a}^{b-1} c_i + \frac{\mathrm{E}(S)\, f'(1)}{b(1-\rho)} \tag{41}$$

where $f'(1) = \alpha\lambda \mathrm{E}(X)\mathrm{E}(S)\sum_{i=a}^{b-1} c_i + \beta\lambda \mathrm{E}(X)\mathrm{E}(V)\sum_{i=0}^{a-1} c_i - \sum_{i=a}^{b-1} ic_i$ and $\rho = \dfrac{\alpha\lambda \mathrm{E}(X)\mathrm{E}(S)}{b}$

5. SPECIAL CASES

The model so developed is general in nature as the service time and vacation time are arbitrary. But for practical purposes, service time and vacation time with particular distribution is required. In this section, some special cases of the proposed model by specifying vacation time random variable as exponential distribution and bulk service time random variable as hyper exponential and Erlangian distributions are discussed.

Case (i): ***Single server batch arrival queue with Hyper Exponential service time and restricted admissibility policy***

Now, the case of hyper exponential service time random variable is considered. The probability density function of hyper exponential service time is given as follows,

$s(x) = cue^{-ux} + (1-c)we^{-wx}$, where u and w are the parameters.

Then, $\tilde{S}(\alpha(\lambda - \lambda X(z))) = \left(\dfrac{uc}{u + \alpha\lambda(1 - X(z))}\right) + \left(\dfrac{w(1-c)}{w + \alpha\lambda(1 - X(z))}\right)$

Hence, the PGF of the queue size distribution of this model can be obtained by,

$$P(z) = \frac{\left\{\begin{aligned} &\beta\sum_{i=a}^{b-1}\left(\left(\left(\frac{uc}{u+\alpha\lambda(1-X(z))}\right)+\left(\frac{w(1-c)}{w+\alpha\lambda(1-X(z))}\right)\right)-1\right)\left(z^b - z^i\right)c_i \\ &+\left(\tilde{V}(\beta(\lambda-\lambda X(z)))-1\right)\sum_{n=0}^{a-1}\left(\begin{aligned}&\beta\left(\left(\left(\frac{uc}{u+\alpha\lambda(1-X(z))}\right)+\left(\frac{w(1-c)}{w+\alpha\lambda(1-X(z))}\right)\right)-1\right)\\ &+\alpha\left(z^b-\left(\left(\frac{uc}{u+\alpha\lambda(1-X(z))}\right)+\left(\frac{w(1-c)}{w+\alpha\lambda(1-X(z))}\right)\right)\right)\end{aligned}\right)c_n z^n\end{aligned}\right\}}{\alpha\beta\left[z^b - \left(\left(\frac{uc}{u+\alpha\lambda(1-X(z))}\right)+\left(\frac{w(1-c)}{w+\alpha\lambda(1-X(z))}\right)\right)\right]\left[\lambda X(z) - \lambda\right]} \tag{42}$$

Case (ii): ***Single server batch arrival queue with Erlangian bulk service time and restricted admissibility policy***

Now, the case of k-Erlang service time random variable is considered. The probability density function of k-Erlang service time is given as follows,

$s(x) = \dfrac{(ku)^k x^{k-1}e^{-kux}}{(k-1)!}$, $k > 0$; where u is the parameter.

Then, $\tilde{S}(\alpha(\lambda - \lambda X(z))) = \left(\dfrac{\mu k}{\mu k + \alpha\lambda(1 - X(z))}\right)^k$.

Hence, the PGF of the queue size distribution of this model can be obtained by,

$$P(z) = \frac{\left\{\begin{aligned} &\beta\sum_{i=a}^{b-1}\left(\left(\frac{\mu k}{\mu k+\alpha\lambda(1-X(z))}\right)^k-1\right)\left(z^b - z^i\right)c_i \\ &+\left(\tilde{V}(\beta(\lambda-\lambda X(z)))-1\right)\sum_{n=0}^{a-1}\left(\beta\left(\left(\frac{\mu k}{\mu k+\alpha\lambda(1-X(z))}\right)^k-1\right)+\alpha\left(z^b-\left(\frac{\mu k}{\mu k+\alpha\lambda(1-X(z))}\right)^k\right)\right)c_n z^n\end{aligned}\right\}}{\alpha\beta\left[z^b - \left(\frac{\mu k}{\mu k+\alpha\lambda(1-X(z))}\right)^k\right]\left[\lambda X(z) - \lambda\right]} \tag{43}$$

Case (iii): ***Single server batch arrival queue with exponential vacation time and restricted admissibility policy***

Now, the case of exponential vacation time random variable is considered. The probability density function of exponential vacation time is given as follows,

$v(x) = \gamma e^{-\gamma x}$, where γ is the parameter.

Then,

$$\tilde{V}(\beta(\lambda - \lambda X(z))) = \left(\frac{\gamma}{\gamma + \beta\lambda(1 - X(z))}\right)$$

Hence, the PGF of the queue size distribution of this model can be obtained by,

$$P(z) = \frac{\left\{\begin{array}{l} \beta\sum_{i=a}^{b-1}\left(\tilde{S}(\alpha(\lambda - \lambda X(z))) - 1\right)\left(z^b - z^i\right)c_i \\ + \left(\left(\frac{\gamma}{\gamma + \beta\lambda(1 - X(z))}\right) - 1\right)\sum_{n=0}^{a-1}\left(\beta\left(\tilde{S}(\alpha(\lambda - \lambda X(z))) - 1\right) + \alpha\left(z^b - \tilde{S}(\alpha(\lambda - \lambda X(z)))\right)\right)c_n z^n \end{array}\right\}}{\alpha\beta\left[z^b - \tilde{S}(\alpha(\lambda - \lambda X(z)))\right]\left[\lambda X(z) - \lambda\right]} \tag{44}$$

5.1 Particular Cases

In this section, some of the existing models as a particular case of the proposed model are derived.

Case (i): If all arrivals are allowed to join the system, i.e. $\alpha = 1$ and $\beta = 1$, then (34) becomes

$$P(z) = \frac{\left\{\sum_{i=a}^{b-1}\left(\tilde{S}(\lambda - \lambda X(z)) - 1\right)\left(z^b - z^i\right)c_i + \left(\tilde{V}(\lambda - \lambda X(z)) - 1\right)\sum_{n=0}^{a-1}\left(z^b - 1\right)c_n z^n\right\}}{\left[z^b - \tilde{S}(\lambda - \lambda X(z))\right]\left[\lambda X(z) - \lambda\right]} \tag{45}$$

which exactly coincides with the result $M^X / G(a,b) / 1$ and multiple vacations without setup time and N – Policy of Krishna Reddy *et al.* (1998).

Case (ii): If all arrivals are allowed to join the system, i.e. $\alpha = 1$, $\beta = 1$ and no bulk service, i.e $a = b = 1$, then (34) becomes

$$P(z) = \frac{\left\{\left(\tilde{V}(\lambda - \lambda X(z)) - 1\right)\left(z - 1\right)c_0\right\}}{\left[z - \tilde{S}(\lambda - \lambda X(z))\right]\left[\lambda X(z) - \lambda\right]} \tag{46}$$

which coincides with the result $M^X / G / 1$ queueing system and multiple vacations without N-Policy of Lee et al.(1994).

Case (iii): Instead of bulk service, if single service is considered (i.e. $a = b = 1$), and all arrivals are allowed to join the system (i.e. $\alpha = 1$ and $\beta = 1$), then the probability that the server is in the busy period and the probability that the server is on vacation at time t is obtained by

$$\begin{aligned} P(B) &= \frac{\mathrm{E}(S)\, f'(1)}{1 - \rho} \\ &= \frac{\mathrm{E}(S)\,\lambda \mathrm{E}(X)\mathrm{E}(V)\, c_0}{1 - \rho} \\ &= \lambda \mathrm{E}(S)\,\mathrm{E}(X) \end{aligned}$$

$= \rho$, is the probability that the server is busy

and $P(V) = \mathrm{E}(V)\, c_0 = 1 - \rho$, is the probability that the system is in vacation period

where the unknown c_0 is obtained from the equation (34) by using the condition $P(1) = 1$, which is $c_0 = \dfrac{1 - \rho}{\mathrm{E}(V)}$, where

$f'(1) = \lambda \mathrm{E}(X)\mathrm{E}(V)\, c_0$ and $\rho = \lambda \mathrm{E}(X)\mathrm{E}(S)$.

This coincides with the result $M^X/G/1$ queueing system with multiple vacations of Sun Hur and Suneung Ahn (2005) without setup times.

6. OPTIMUM COST

Cost analysis is the most important phenomenon in any practical situation at every stage. Cost involves startup cost, operating cost, holding cost and reward cost (if any). It is quite natural that the management of the system desires to minimize the total average cost and optimize the cost. Addressing this, in this section, the cost model for the proposed queueing system is developed and the total average cost is obtained with the following assumptions:

C_s : Startup cost per cycle

C_h : Holding cost per customer

C_o : Operating cost per unit time

C_r : Reward cost per cycle due to vacation

Since the length of the cycle is the sum of the idle period and busy period, from the equations (37) and (38), the expected length of cycle, $\mathrm{E}(T_c)$ is obtained as

E (T_c) = E (length of the Idle Period)+E(length of the Busy Period)

$$= \frac{\mathrm{E}(V)}{1 - \sum_{n=0}^{a-1}\sum_{i=0}^{n}\omega_i p_{n-i}} + \frac{\mathrm{E}(S)}{\sum_{i=0}^{a-1} p_i}$$

Now, the total average cost per unit time is obtained as,

Total average cost = Start-up cost per cycle + holding cost of number of customers in the queue per unit time + Operating cost per unit time * ρ – reward due to vacation per cycle.

$$=\left[C_s - C_r \mathrm{E}(I) \right]\frac{1}{\mathrm{E}(T_c)} + C_h \mathrm{E}(Q) + C_o\, \rho \tag{47}$$

where $\rho = \dfrac{\alpha\lambda \mathrm{E}(X)\mathrm{E}(S)}{b}$.

It is difficult to have a direct analytical result for the optimal value a^* (minimum batch size in $M^X/G(a,b)/1$ queueing system) to minimize the total average cost. The simple direct search method to find optimal policy for a threshold value a^* to minimize the total average cost, is defined.

Step 1: Fix the value of maximum batch size '*b*'

Step 2: Select the value of '*a*' which will satisfy the following relation

$$\mathrm{TAC}\left(a^*\right) \le \mathrm{TAC}(a), \qquad 1 \le a \le b$$

Step 3: The value a^* is optimum, since it gives minimum total average cost.

Using the above procedure, the optimal value of '*a*' can be obtained, which minimizes the total average cost function. Some numerical example to illustrate the above procedure is presented in the next section.

7. NUMERICAL ILLUSTRATION

In this section, a numerical example is analyzed to illustrate how the management of an electroplating processing system can effectively use the results obtained in the sections 3 – 4 to take decision regarding effectively utilizing the idle time and to identify the threshold value to minimize the total average cost.

In the electro plating process centre, the arrival of components (customers) follow Poisson process with arrival rate λ, the process of the components is done in bulk (bulk service). Once the process is started, the bulk operation has to continue successively for many batches of components, otherwise, the operating cost will increase. Hence, the operator will start the electroplating process only when required numbers of pieces have been accumulated for processing (bulk service). After completing an electro plating process, if the number of pieces to be processed is less than the batch quantity, say '*a*', then, the operator stops the process and performs the associated works (vacation), such as rinsing, unjigging the components, buffing, inspection, etc., Further, in order to meet the customer satisfaction and to deliver the processed electroplates in time, the management may reject new order (arrivals) with some probability. The operator accepts only 'α' percent of arriving batch when the server is busy and 'β' percent of arriving batch when the server is on vacation. Addressing this, the consistencies of the theoretical results obtained in the sections 3 – 4 are justified numerically with the following assumptions and notations:

Service time distribution is 2- Erlang with parameter	μ
Batch size distribution of the arrival is geometric with mean	2
Vacation time is exponential with parameter	γ
Minimum service capacity	a
Maximum service capacity	b
Probability of arriving batch will be allowed to join the system during the busy period	α
Probability of arriving batch will be allowed to join the system during the vacation period	β

7.1 Effects of Various Parameters on the Performance Measures

The effects of various parameters such as arrival rates, expected queue length, expected idle period, expected busy period, probability that the server is on vacation, probability that the server is busy, different probabilities of admitting arrivals to join the system during busy period, during the vacation period and threshold value '*a*' are analyzed numerically and presented in tables 1 – 4 and represented in Figures 1 – 3. All numerical results are obtained using Mat Lab software.

The effects of various performance measures for a fixed '*a*' and '*b*' with respect to different probabilities of admitting arrivals to join the system during busy period are obtained numerically. These results are tabulated in table 1. It is observed that, if the probability of admitting customers during busy period increases, then

(1) the expected queue length, the expected busy period and the probability that the server is busy increase
(2) the probability that the server is on vacation and the expected idle period decrease.

In table 2, for different arrival rates, the effects of various performance measures for a fixed '*a*' and '*b*' are presented. From the table, it is clear that, if the arrival rate increases, then

(1) the mean queue size, the mean busy period and the probability that the server is busy increase
(2) the mean idle period and the probability that the server is on vacation decrease.

The effects of threshold value '*a*' on the expected queue length for various probabilities of admitting customers during the busy period are obtained numerically, and these results are tabulated in table 3. From the table, the following observations are made:

(1) For a fixed threshold value '*a*', when the probability of allowing customers during busy period increases, the expected queue length increases.
(2) For a fixed probability (α) of allowing customers during busy period, when threshold value increases, the expected queue length increases.

The effects of threshold value '*a*' on the expected queue length for various probabilities of admitting customers during the vacation period are obtained numerically, and these results are tabulated in table 4. From the table, the following observations are made:

(1) For a fixed threshold value '*a*', when the probability of allowing customers during vacation period increases, the expected queue length increases.
(2) For a fixed probability (β) of allowing customers during vacation period, when threshold value increases, the expected queue length increases.

7.2 Optimal Cost

In this section, a numerical example is analyzed to illustrate how the management of an electroplating processing system can effectively use the results obtained in the sections 3,4 and 6, to make the decision regarding the threshold value to minimize the total average cost.

It is assumed that, the maximum capacity of an electroplating process is 12 units (i.e. b = 12 pieces). If the management of an electroplating process allows the operator to start the process even for a single piece (i.e. $a = 1$) without waiting for further arrival, clearly, the operating cost will increase. On the other hand, if they start the process until all 12 pieces arrive, the holding cost may increase; hence, there must be some value between 1 and 12 that will optimize the cost. An optimal policy regarding the threshold value '*a*' which will minimize the total average cost is wished to be obtained.

The total average costs are obtained numerically with the following assumptions:

Startup cost	: 4.00
Holding cost per customer	: 0.25
Operating cost per unit time	: 7.00
Reward cost per unit time due to vacation	: 1.00

The effects of the threshold value '*a*' on the total average cost with b = 12 are reported in tables 3 and 4, and represented in Figures 1(a-b), 2 and 3.

From the table 3 and the Figure 2, it is clear that, for an electroplating process center with the capacity of 12 pieces (i.e. b = 12) at a time, the *management has to fix the threshold value a = 5 to minimize the total average cost* for the probability of admitting pieces during the busy period 0.2 (i.e. $\alpha = 0.2$) and the probability of admitting pieces during the vacation period 0.6 (i.e. $\beta = 0.6$). From the table 4 and the Figures 1 and 3, the *management has to fix the threshold value a = 4 to minimize the total average cost* for the probability of admitting pieces during the busy period 0.7 (i.e. $\alpha = 0.7$) and the probability of admitting pieces during the vacation period 0.2 (i.e. $\beta = 0.2$).

Similarly, the management has to fix the threshold value '*a*' to minimize the total average cost for various probabilities of admitting pieces during the vacation and non-vacation periods.

Table 1. Probability of admitting customers during busy period (*Vs*) performance measures

(For λ =3.5; μ =2.0; a = 3; b = 4; γ =15; β =0.9)

α	P(*B*)	P(*V*)	E(*Q*)	E(*B*)	E(*I*)
0.2	0.5201	0.4799	1.4585	0.6380	0.2336
0.4	0.5833	0.4167	2.0167	0.7082	0.1845
0.6	0.6608	0.3392	3.0514	0.8455	0.1423
0.8	0.7582	0.2418	5.5017	1.1587	0.1086
1.0	0.8843	0.1157	15.6384	2.3754	0.0820

Table 2. Arrival rate (*Vs*) performance measures
(For μ =2.0; a = 3; b = 4; γ =15; α =0.8; β =0.7)

λ	P(*B*)	P(*V*)	E(*Q*)	E(*B*)	E(*I*)
2.5	0.5098	0.4902	2.3981	1.0200	0.1241
3.0	0.6109	0.3891	3.4015	1.0548	0.1192
3.5	0.7107	0.2893	5.1706	1.2186	0.1069
4.0	0.8090	0.1910	8.8425	1.6153	0.0925
4.5	0.9055	0.0945	20.0994	2.8992	0.0788

P(*V*) - Probability that the server is on multiple vacations; P(*B*) - Probability that the server is busy; E(*Q*) – Expected queue length; E(*I*) – Expected idle period; E(*B*) – Expected busy period

Table 3. Threshold value (*Vs*) expected queue length and total average cost for different probabilities (α) of allowing customers during busy period
(For λ =1.0, μ =2.5, b=12, γ =2, β =0.6)

a	E(*Q*)					TAC				
	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1.0$	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1.0$
1	0.8913	0.9407	0.9947	1.0518	1.3517	2.1966	2.2942	2.3922	2.4907	2.5895
2	1.0484	1.0911	1.1355	1.1790	1.4665	2.0023	2.1092	2.2158	2.3205	2.4253
3	1.3468	1.3913	1.4351	1.4765	1.6552	1.8844	1.9990	2.1120	2.2225	2.3320
4	1.7148	1.7630	1.8078	1.8505	1.8924	1.8170	1.9362	2.0521	2.1664	2.2805
5	2.1247	2.1751	2.2220	2.2671	2.3233	**1.7881**	**1.9080**	**2.0256**	**2.1417**	2.2465
6	2.5631	2.6165	2.6640	2.7121	2.7762	1.7888	1.9091	2.0258	2.1430	**2.2429**
7	3.0257	3.0808	3.1294	3.1801	3.2890	1.8148	1.9342	2.0499	2.1676	2.2628
8	3.5133	3.5688	3.6197	3.6717	3.6827	1.8628	1.9804	2.0964	2.2131	2.3042
9	4.0305	4.0870	4.1406	4.1920	4.2740	1.9310	2.0479	2.1643	2.2790	2.3667
10	4.5917	4.6487	4.7045	4.7572	4.8379	2.0226	2.1388	2.2551	2.3698	2.4540
11	5.2259	5.2830	5.3401	5.3939	5.4766	2.1459	2.2614	2.3775	2.4921	2.5733
12	5.9864	6.0440	6.1007	6.1558	6.1873	2.3156	2.4310	2.5463	2.6611	2.7403

E(*Q*) – Expected queue length; TAC – Total average cost

Table 4. Threshold value (Vs) expected queue length and total average cost

for different probabilities (β) of allowing customers during vacation
(For λ=1.0, μ=2.5, b=12, γ=2, α=0.7)

a	E(Q)					TAC				
	$\beta=0.2$	$\beta=0.4$	$\beta=0.6$	$\beta=0.8$	$\beta=1.0$	$\beta=0.2$	$\beta=0.4$	$\beta=0.6$	$\beta=0.8$	$\beta=1.0$
1	0.4283	0.7564	1.0221	1.2469	1.4437	1.5730	2.1389	2.4414	2.6359	2.7797
2	0.6806	0.9345	1.1565	1.3557	1.5381	1.3940	1.9425	2.2680	2.4842	2.6361
3	1.0625	1.2676	1.4553	1.6299	1.7948	1.3267	1.8363	2.1671	2.3995	2.5651
4	1.4964	1.6676	1.8292	1.9840	2.1335	**1.3219**	1.7858	2.1095	2.3519	2.5291
5	1.9566	2.1032	2.2454	2.3847	2.5220	1.3519	**1.7737**	**2.0844**	**2.3308**	**2.5150**
6	2.4327	2.5616	2.6899	2.8180	2.9464	1.4054	1.7908	2.0857	2.3332	2.5205
7	2.9195	3.0372	3.1566	3.2778	3.4009	1.4751	1.8298	2.1099	2.3559	2.5436
8	3.4198	3.5318	3.6471	3.7656	3.8870	1.5591	1.8885	2.1554	2.3990	2.5868
9	3.9380	4.0505	4.1668	4.2869	4.4106	1.6556	1.9654	2.2217	2.4633	2.6512
10	4.4907	4.6089	4.7311	4.8571	4.9866	1.7685	2.0654	2.3125	2.5532	2.7417
11	5.1053	5.2344	5.3670	5.5025	5.6407	1.9036	2.1900	2.4347	2.6757	2.8657
12	5.8415	5.9841	6.1282	6.2738	6.4209	2.0766	2.3597	2.6037	2.8462	3.0380

E(Q) – Expected queue length; TAC – Total average cost

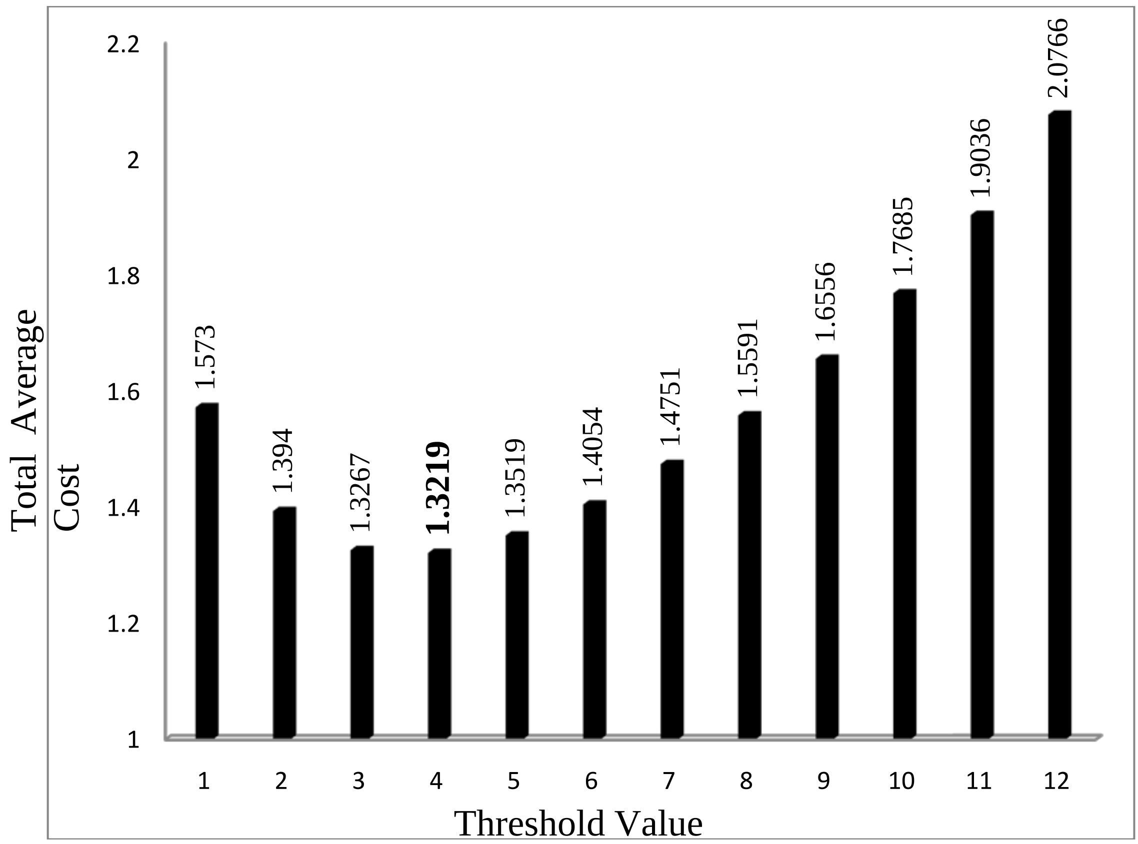

Figure 1. (a) Threshold Value (Vs) Total Average Cost for α=0.7, β=0.2
(For λ=1.0, μ=2.5, b=12, γ=2)

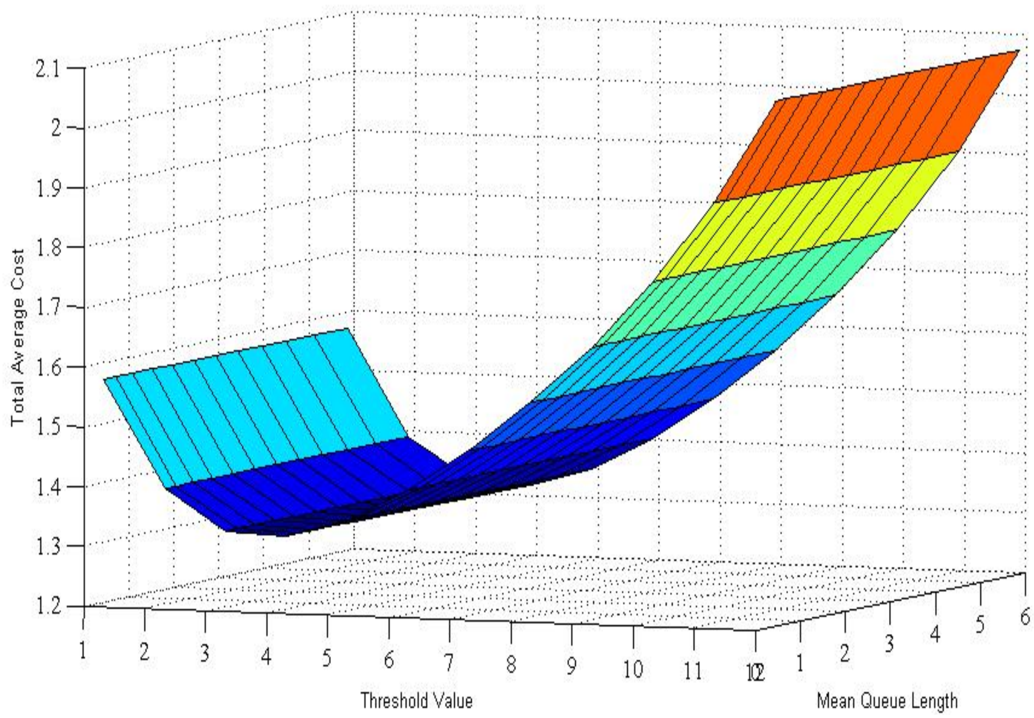

Figure 1. (b) Threshold value (Vs) total average cost for α =0.7, β =0.2
(For λ =1.0, μ =2.5, b=12, γ =2)

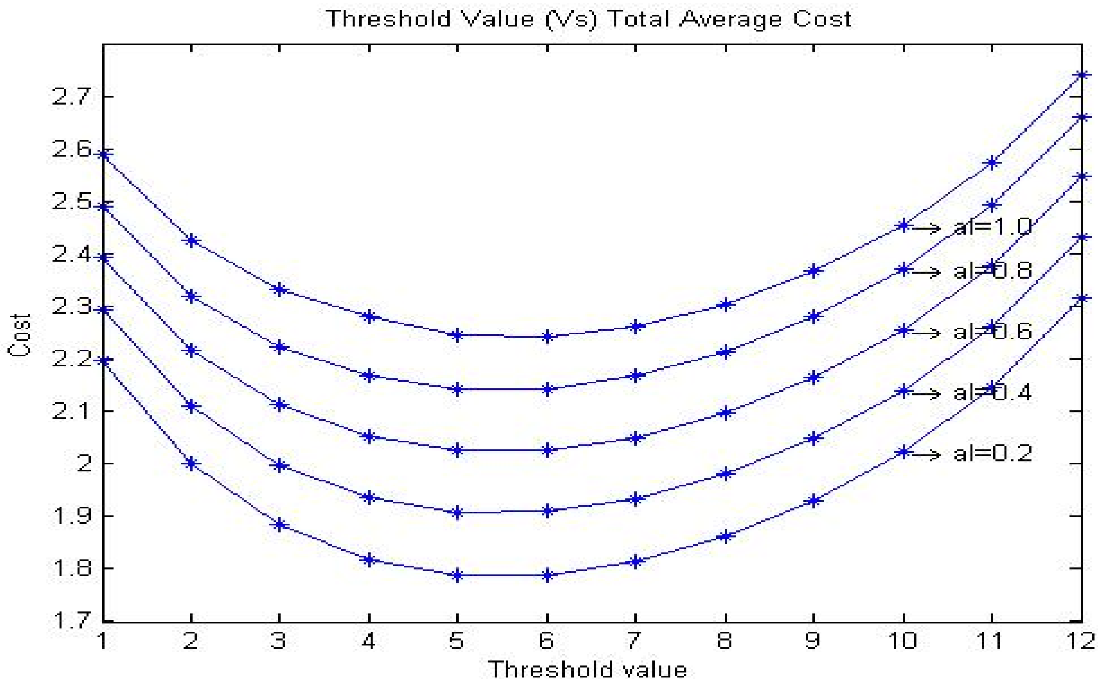

Figure 2. Threshold value (Vs) total average cost for different
probabilities (α) of allowing customers during busy period
(For λ =1.0, μ =2.5, b=12, γ =2, β =0.6)

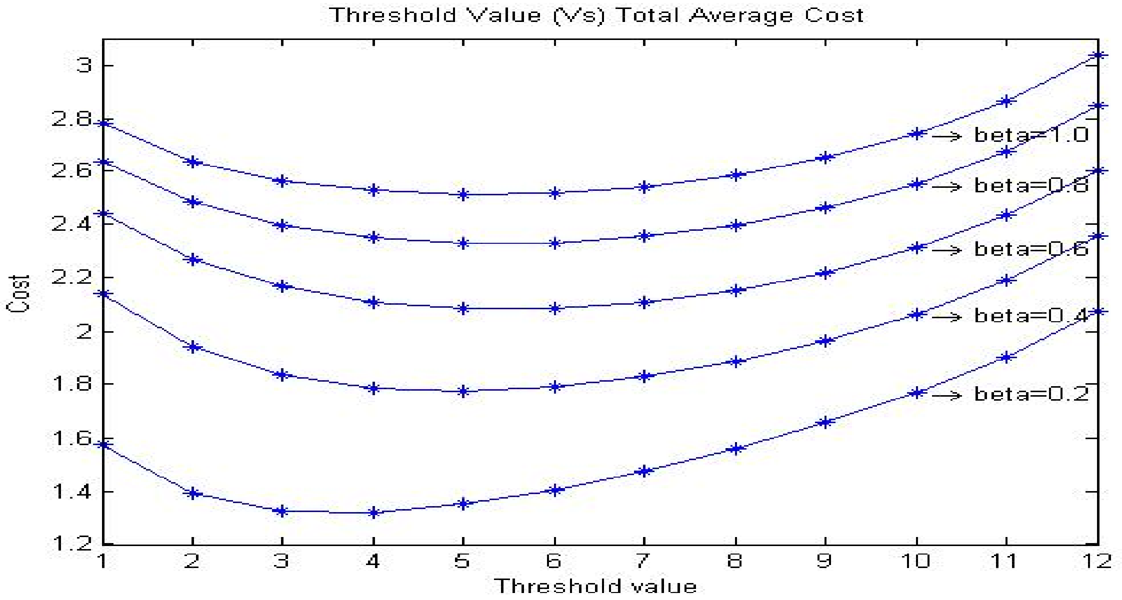

Figure 3. Threshold value (Vs) total average cost for different probabilities (β) of allowing customers during vacation (For λ=1.0, μ=2.5, b=12, γ=2, α=0.7)

8. CONCLUSION

In this paper, "a batch arrival general bulk service queueing system under a restricted admissibility policy of arriving batches with multiple vacations" is analyzed. The probability generating function for the queue size at an arbitrary time epoch is derived. Various performance measures are also obtained. The results so obtained in this paper can be used for managerial decision to optimize the overall cost and search for the best operating policy in a waiting line system. The theoretical development of the model is justified with numerical results which are consistent with the fact that the total average cost decreases when the restricted admissibility policy is adopted during the vacation and non-vacation periods (busy).

In the direction of future research, the model can be extended with service interruptions, close down concepts. An attempt may be made to derive the busy period distributions and idle period distributions. A discrete time model can also be developed.

ACKNOWLEDGEMENT

The authors would be grateful to thank the referees and the editor for their insightful comments and suggestions, which helped in bringing the paper to its present form.

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