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A Multi-Server Markovian Queueing System with Discouraged Arrivals and Retention of Reneged Customers

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Abstract — Customer impatience (reneging) has a very negative impact on the queueing systems. If we talk from business point of view, the firms lose their potential customers due to customer impatience which affects their business as a whole. If the firms employ certain customer retention strategies then there are chances that a certain fraction of impatient customers can be retained in the queueing system. A reneged customer may convinced to stay in the queueing system for his further service with some probability, say q and he may abandon the queue without receiving the service with probability $p(=1-q)$. In this paper, we study a multi-server, finite capacity Markovian queueing model with discouraged arrivals, reneging, and retention of reneged customers. The steady-state probabilities of system size are derived explicitly. Some useful measures of effectiveness are derived and discussed. The cost-profit analysis of the model is performed. Further, the effect of discouraged arrivals on the expected system size is studied. Some comparison of this model with the model by Kumar and Sharma (2013) are also performed. In order to study the effect of discouraged arrivals on the total expected profit of the system. Finally, some important queueing models are derived as the special cases of the model..

Keywords — Probability of customer retention, reneging, discouraged arrivals, steady-state solution, measures of effectiveness, cost model

1. INTRODUCTION

Queueing theory plays an important role in modelling real life problems involving congestions in wide areas of science, technology and management. Applications of queueing with customer impatience can be seen in traffic modelling, business and industries, computer-communication, health sectors and medical sciences etc. Queueing systems with extensions like retrial, feedback, impatience, discouragement etc. are extensively studied by the queueing modelers to demonstrate the special activity performed by the customers. All these extensions in standard queueing systems have their own impact on the system performance.

Queues with discouraged arrivals have applications in computers with batch job processing where job submissions are discouraged when the system is used frequently and arrivals are modelled as a Poisson process with state dependent arrival rate. The discouragement affects the arrival rate of the queueing system. Morse (1968) considers discouragement in which the arrival rate falls according to a negative exponential law. We consider c servers and the customers arrive into a multi-server queueing system in a Poisson fashion with rate λ and a customer finding every server busy arrive with arrival rate that depends on the number of customers present in the system at that time i.e. if there are n (n>c) customers in the system, the new customer enters the system with rate $\dfrac{\lambda}{(n-c)+1}$.

Queueing with customer impatience has vast applications in computer-communications, bio- medical modelling, service systems etc. It is important to note that the prevalence of the phenomenon of customer impatience has a very negative impact on the queueing system under investigation. If we talk from business point of view, the firms lose their potential customers due to customer impatience which affects the business of firms as a whole. If firms employ certain customer retention strategies then there are chances that a certain fraction of impatient customers can be retained in the queueing system. An impatient customer (due to reneging) may be convinced to stay in service system for his service by utilizing certain convincing mechanisms. Such customers are termed as retained customers. When a customer gets impatient (due to reneging), he may leave the queue with some probability, say p and may remain in the queue for service with probability $q(=1-p)$.

Taking these concepts into consideration, a multi-server, finite capacity queueing model with discouraged arrivals, reneging and retention of reneged customers is developed. The steady-state solution of the model is derived. The

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cost-profit analysis of the model is also carried out. Some important queueing models are obtained as special cases of this model.

Rest of the paper is structured as follows: In section 2, literature review is presented. In section 3, we describe the queueing model. The differential-difference equations are derived and solved iteratively in section 4. Measures of effectiveness are derived and discussed in section 5. In section 6, the cost model is presented and some comparisons are performed. The special cases of the model are derived in section 7. The conclusions and future work are provided in section 8.

2. LITERATURE REVIEW

Queueing with impatience finds its origin during the early 1950's. Haight (1959) studies a single-server Markovian queueing system with reneging. Ancker and Gafarian (1963a) study $M/M/1/N$ queueing system with balking and reneging, and perform its steady state analysis. Ancker and Gafarian (1963b) also obtain results for a pure balking system (no reneging) by setting the reneging parameter equal to zero. Multi-server queueing systems with customer impatience find their applications in many real life situations such as in hospitals, computer-communication, retail stores etc. Montazer-Haghighi *et al.* (1986), Abou-El-Ata and Hariri (1992), Falin and Artalejo (1995), Boots and Tijms (1999), and Zohar *et al.* (2002) study and analyze some multi-server queueing systems with balking and reneging. Liu and Kulkarni (2008) consider the virtual queueing time process in an M/G/s queue with impatient customers. They focus on the virtual queueing time based balking model and relate it to reneging behaviour of impatient customers in terms of the steady-state distribution of the virtual queueing time process. Altman and Yechiali (2008) study an infinite- server queue with system's additional tasks and impatient customers. Shin and Choo (2009) consider an M/M/s queue with impatient customers and retrials. The customer who is balked at entering the system or reneged on waiting line can join the virtual pool of customers, called orbit and repeat its request after random amount of time. They describe the number of customers in orbit and service facility by a Markov chain on two-dimensional lattice space and obtain its stationary distribution. Al-seedy et al. (2009) study M/M/c queue with balking and reneging and derive its transient solution by using the probability generating function technique and the properties of Bessel function. Xiong and Altiok (2009) study multi-server queues with deterministic reneging times with reference to the timeout mechanism used in managing application servers in transaction processing environments. Wang *et al.* (2010) present an extensive review on queueing systems with impatient customers.

Choudhury and Medhi (2010) study customer impatience in multi-server queues. They consider both balking and reneging as functions of system state by taking into consideration the situations where the customer is aware of its position in the system. Kapodistria (2011) study a single server Markovian queue with impatient customers and considered the situations where customers abandon the system simultaneously. Kumar (2012) investigates a correlated queueing problem with catastrophic and restorative effects with impatient customers which have special applications in agile broadband communication networks. Kumar and Sharma (2012a) study an M/M/1/N queue with balking and retention of reneged customers. They study its application in supply chain and discussed the cost-profit aspects of the model.

Queueing models where potential customers are discouraged by queue length are studied by many researchers in their research work. Natvig (1975) studies the single server birth-death queueing process with state dependent parameters $\lambda_n = \dfrac{\lambda}{n+1}, n \geq 0$ and $\mu_n = \mu \; n \geq 1$. He reviews state dependent queueing models of different kinds and compare his results with M/M/1, M/D/1, D/M/1 and the single server birth-and-death queueing model with parameters $\lambda_n = \lambda, n \geq 0$ and $\mu_n = n\mu \,, n \geq 1$. Raynolds (1968) studies multi-server queueing model with discouragement. He obtains equilibrium distribution of queue length and derives other performance measures. Cuortois and Georges (1971) study finite capacity M/G/1 queueing model where the arrival and the service rates are arbitrary functions of the number of customers in the system. They obtain results for expected value of time needed to complete a service including waiting time distribution. Hadidi (1974) carries out analysis of busy period processes for $M/M_n/1$ and $M_n/M/1$ queueing models with state dependent service and arrival rates. He also obtains results for busy period and transient state probabilities. Von Doorn (1981) obtains exact expressions for transient state probabilities of the birth death process with parameters $\lambda_n = \dfrac{1}{n+1}\lambda, n \geq 0$ and $\mu_n = \mu, n \geq 1$. Ammar *et al.* (2012) study single server, finite capacity Markovian queue with discouraged arrivals and reneging and obtain the transient solution of the model by using matrix method.

Parathasarathy and Lenin (1998) obtain transient solution of a state-dependent multi-server queueing model by using continued fractions approach. They also derived the busy period of the model. Parathasarathy *et al.* (2000) study multiprocessor systems with state dependent arrivals using queueing theory and obtain the transient solution. Parathasarathy and Sudhesh (2005) express inter-overflow time distribution as a power series expansion in closed form and as a hyper-exponential distribution in closed from for a finite capacity queueing system with general state dependent arrival and service rate. Further, they obtain a closed form expression for the r^{th} moment of the given finite state overflow process. El-Paoumy and Nabwey (2011) analyse M/M/2/N queueing model with balking function, reneging, and heterogeneous server and obtain the steady-state solution. Kumar and Sharma (2012b) obtain the transient solution of an M/M/c/N queueing model with balking, reneging and retention of reneged customers. They also perform the economic analysis of the model. Kumar and Sharma (2012c) study a Markovian single-server finite capacity queueing system with retention of reneged customers and balking. They obtain the steady-state results and perform sensitivity analysis of the model. Jouini (2012) study a single class multi-server queueing system with abandonment and LSFS discipline. Recently, Kumar and Sharma (2013) study an M/M/c/N queueing model with customer retention. They incorporate the probability of retaining the reneged customers in their model. They perform the steady-state analysis of the model and also study the effect of probability of retaining the reneged customers on expected system size.

In the present paper, we study a finite capacity multi-server Markovian queue with discouraged arrivals and retention of reneged customers.

3. QUEUEING MODEL DESCRIPTION

In this section, we describe the queueing model. The review of literature shows that various extensions are made with reference to reneging, balking and discouragement concepts in queueing modelling. However, in the current competitive era, the business firms are constantly striving for the retention of impatient customers. Keeping in mind this practically valid aspect, an endeavour is made in this paper to study the retention of impatient (reneged) customers in a multiserver Markovian queue with discouraged arrivals.

The customers arrive to the queueing system according to a Poisson process with parameter λ. A customer finding every server busy arrive with arrival rate that depends on the number of customers present in the system at that time i.e. if there are *n* (*n*>*c*) customers in the system, the new customer enters the system with rate $\frac{\lambda}{(n-c)+1}$.There are *c* servers and the service times at each server are independently, identically and exponentially distributed with parameter μ. The mean service rate is given by $\mu_n = \left\{ n\mu; 0 \le n \le c-1 \,\&\, c\mu; c \le n \le N \right\}$. The customers are served in order of their arrival, that is, the queue discipline is FCFS. The capacity of the system is finite (say, N). That is, the system can accommodate at most N customers. A queue gets developed when the number of customers exceeds the number of servers, that is, when $n > c$. Each customer upon joining the queue will wait a certain length of time for his service to begin. If it has not begun by then, he will get impatient (reneged) and may leave the queue without getting service with probability p and may remain in the queue for his service with probability $q(= 1 - p)$. The reneging times follow exponential distribution with parameter ξ.

4. DIFFERENTIAL DIFFERENCE EQUATIONS AND SOLUTION OF THE QUEUEING MODEL

In this section, the mathematical framework of the queueing model is presented. Let $P_n(t)$ be the probability that there are n customers in the system at time *t*. The differential-difference equations are derived by using the general birth-death arguments. These equations are solved iteratively in steady-state in order to obtain the steady state solution.

The differential-difference equations of the model are:

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1 \tag{1}$$

$$\frac{dP_n(t)}{dt} = -(\lambda + n\mu) P_n(t) + ((n+1)\mu) P_{n+1}(t) + \lambda P_{n-1}(t); 1 \le n \le c-1 \tag{2}$$

$$\frac{dP_n(t)}{dt} = -\left[\left(\frac{\lambda}{n-c+2}\right) + c\mu + (n-c)\xi p\right] P_n(t) + \left[c\mu + \left\{(n+1) - c\right\}\xi p\right] P_{n+1}(t) + \left(\frac{\lambda}{n-c+1}\right) P_{n-1}(t); c \le n \le N-1 \tag{3}$$

$$\frac{dP_N(t)}{dt} = \left(\frac{\lambda}{N-c+1}\right) P_{N-1}(t) - \left\{c\mu + (N-c)\xi p\right\} P_N(t); n = N \tag{4}$$

In steady-state, $\lim_{n\to\infty} P_n(t) = P_n$ .Therefore, the steady-state equations corresponding to equations (1) - (4) are as follows:

$$0 = -\lambda P_0 + \mu P_1 \tag{5}$$

$$0 = -(\lambda + n\mu) P_n + ((n+1)\mu) P_{n+1} + \lambda P_{n-1}, 1 \le n \le c-1` \tag{6}$$

$$0=-\left[\left(\frac{\lambda}{n-c+2}\right)+c\mu+\left(n-c\right)\xi p\right]P_n+\left[c\mu+\left\{\left(n+1\right)-c\right\}\xi p\right]P_{n+1}+\left(\frac{\lambda}{n-c+1}\right)P_{n-1};\, c\le n\le N-1 \tag{7}$$

$$0=\left(\frac{\lambda}{N-c+1}\right)P_{N-1}-\left\{c\mu+\left(N-c\right)\xi p\right\}P_N;\, n=N \tag{8}$$

Solving the equations (5) – (8) iteratively, we obtain

$$P_n=\begin{cases}\dfrac{\lambda^n}{n!\,\mu^n}P_0;1\le n\le c\\[2ex] \dfrac{1}{(n-c+1)!}\displaystyle\prod_{k=c+1}^{n}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{\lambda^c}{c!\,\mu^c}P_0;c+1\le n\le N\end{cases} \tag{9}$$

Using the normalization condition, $\sum_{n=0}^{N}P_n=1$, we get

$$P_0=\frac{1}{\left(1+\sum_{n=1}^{c}\frac{\lambda^n}{n!\,\mu^n}+\sum_{n=c+1}^{N}\frac{1}{(n-c+1)!}\prod_{k=c+1}^{n}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{\lambda^c}{c!\,\mu^c}\right)}. \tag{10}$$

Hence, the steady-state probabilities of the system size are derived explicitly.

5. MEASURES OF EFFECTIVENESS

In this section, some important measures of effectiveness are derived. These can used to study the performance of the queueing system under consideration.

The Expected System Size (Ls):
The expected number of customers in the system is given as:

$$L_s=\left[\sum_{n=1}^{c}n\frac{\lambda^n}{n!\,\mu^n}+\sum_{n=c+1}^{N}n\frac{1}{(n-c+1)!}\prod_{k=c+1}^{n}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{\lambda^c}{c!\,\mu^c}\right]P_0$$

The Expected Queue Length (Lq):
The expected number of customers in the queue is given as:

$$L_q=\left[\sum_{n=1}^{c}n\frac{\lambda^n}{n!\,\mu^n}+\sum_{n=c+1}^{N}n\frac{1}{(n-c+1)!}\prod_{k=c+1}^{n}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{\lambda^c}{c!\,\mu^c}\right]-\frac{\lambda}{\mu}$$

The Expected Waiting Time in the System (Ws):
The expected number of customers waiting in the system is given as:

$$W_s=\left\{\left[\sum_{n=1}^{c}n\frac{\lambda^n}{n!\,\mu^n}+\sum_{n=c+1}^{N}n\frac{1}{(n-c+1)!}\prod_{k=c+1}^{n}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{\lambda^c}{c!\,\mu^c}\right]P_0\right\}/\lambda$$

The Expected Waiting Time in the Queue (Wq):
The expected number of customers waiting in the queue is given as:

$$W_q=\frac{\left\{\left[\sum_{n=1}^{c}n\frac{\lambda^n}{n!\,\mu^n}+\sum_{n=c+1}^{N}n\frac{1}{(n-c+1)!}\prod_{k=c+1}^{n}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{\lambda^c}{c!\,\mu^c}\right]P_0\right\}}{\lambda}-\frac{1}{\mu}$$

The Expected Number of Customers Served, E(Customer Served):
The expected number of customers served is given by:

$$\text{E(Customer Served)}=\sum_{n=1}^{c}n\mu P_n+\sum_{n=c+1}^{N}c\mu P_n$$

$$\text{E(Customer Served)}=\left[\sum_{n=1}^{c}n\mu\frac{\lambda^n}{n!\,\mu^n}+\sum_{n=c+1}^{N}c\mu\frac{1}{(n-c+1)!}\prod_{k=c+1}^{n}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{\lambda^c}{c!\,\mu^c}\right]P_0$$

Rate of Abandonment, R_{aband}:
The average rate at which the customers abandon the system is given by:

$$R_{\text{aband}} = \lambda \sum_{n=0}^{N} P_n \text{ - E(Customer Served)}$$

$$R_{\text{aband}} = \lambda - \left[\sum_{n=1}^{c} n\mu \frac{\lambda^n}{n!\,\mu^n} + \sum_{n=c+1}^{N} c\mu \frac{1}{(n-c+1)!} \prod_{k=c+1}^{n} \frac{\lambda}{c\mu + (k-c)\xi p} \frac{\lambda^c}{c!\,\mu^c} \right] P_0$$

Expected number of waiting customers, who actually wait, E(Actual Cust. Waiting):
The Expected number of waiting customers, who actually wait is given by:

$$\text{E(Actual Cust. Waiting)} = \frac{\sum_{n=c+1}^{N} (n-c) P_n}{\sum_{n=c+1}^{N} P_n} = \frac{\sum_{n=c+1}^{N} (n-c) \left[\frac{1}{(n-c+1)!} \prod_{k=c+1}^{n} \frac{\lambda}{c\mu + (k-c)\xi p} \frac{\lambda^c}{c!\,\mu^c} \right] P_0}{\sum_{n=c+1}^{N} \left[\frac{1}{(n-c+1)!} \prod_{k=c+1}^{n} \frac{\lambda}{c\mu + (k-c)\xi p} \frac{\lambda^c}{c!\,\mu^c} \right] P_0}$$

5.1. Variation in expected system size with the change in probability of customer retention (a comparison of two models: M/M/c/N queue with customer retention and discouragement, and M/M/c/N queue with customer retention)

In section 5, various measures of effectiveness are derived. In this sub-section, we compare two queueing models viz. M/M/c/N queueing model with customer retention as studied by Kumar and Sharma (2013) and M/M/c/N queueing model with customer retention and discouragement with respect to the effect of probability of customer retention on expected system size. The numerical results are computed by using MS-Excel and shown in table-1. It can be observed that the expected system size increases with increase in probability of customer retention in both the cases, but it remains always higher in the case of M/M/c/N queue with customer retention than that of M/M/c/N queue with customer retention and discouragement. Thus, discouragement has negative impact on the expected system size.

Table 1. Variation in expected system size with the change in probability of customer retention (a comparison of two models: M/M/c/N queue with customer retention and discouragement, and M/M/c/N queue with customer retention)
When $N = 4, c = 2, \lambda = 2, \mu = 3, \xi = 0.1$

S.No.	Probability of retaining the reneged customers (q)	M/M/c/N queue with discouragement and customer retention Ls	M/M/c/N queue with discouragement and customer retention Ls
1	0	0.557633	0.71981
2	0.05	0.557692	0.759766
3	0.1	0.55775	0.800024
4	0.15	0.557809	0.840551
5	0.2	0.557867	0.881313
6	0.25	0.557926	0.922272
7	0.3	0.557985	0.963391
8	0.35	0.558044	1.004628
9	0.4	0.558103	1.45946
10	0.45	0.558162	1.087301
11	0.5	0.558222	1.128655
12	0.55	0.558281	1.169967
13	0.6	0.558341	1.211197
14	0.65	0.558401	1.252305
15	0.7	0.558461	1.293255
16	0.75	0.558521	1.33401
17	0.8	0.558581	1.374533
18	0.85	0.558642	1.414793
19	0.9	0.558702	1.454756
20	0.95	0.558763	1.494393
21	1	0.558824	1.516427

IJOR Vol. 9, No. 4, 173–184 (2012)

5.2. Comparison of M/M/c/N queue with customer retention, and M/M/c/N queue with customer retention and discouragement with respect to mean arrival rate

In this sub-section, we compare M/M/c/N queue with customer retention and discouragement with that of M/M/c/N queue with customer retention with respect to the change in expected system size with mean arrival rate. It is clear from figure-1 that the expected system size remains always lower in case of M/M/c/N queue with customer retention and discouragement as compared to M/M/c/N queue with customer retention. In this way one can study the adverse effect of discouragement on expected system size.

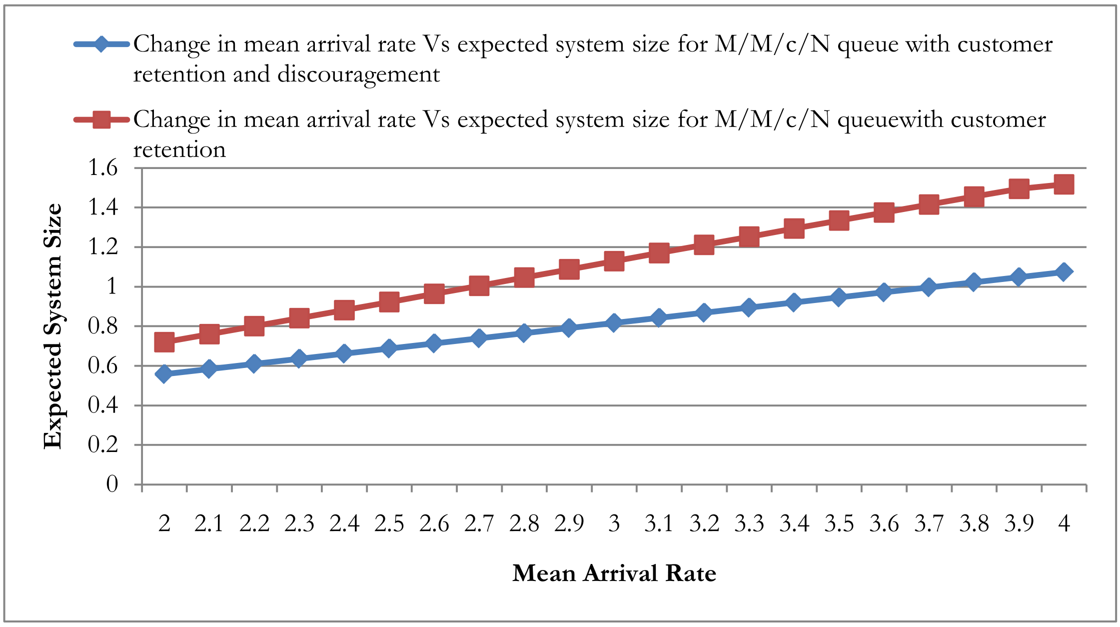

Figure 1. Change in expected system size with respect to mean arrival rate

6. COST MODEL

In this section the cost model is presented. The cost-profit analysis of the model is performed. The numerical results are obtained by translating the cost and profit functions in MS-Excel. The effect of probability of customer retention on total expected cost and total expected profit is discussed. Moreover, we study and compare two queueing models viz. M/M/c/N queueing model with customer retention, and M/M/c/N queueing model with customer retention and discouragement with respect to the effect of probability of customer retention on total expected cost, total expected revenue and total expected profit.

The different parameters involved are given below:

$\frac{1}{\lambda}$ = Mean inter arrival time; $\frac{1}{\mu}$ = Mean service time

λ_{lost} =Rate at which customer would be lost

P_N = Probability that the system is full

L_s = Expected number of customer in the system

R_r = Average rate of reneging; R_r = Average rate of retention

C_S = Cost per service per unit time

C_h = Holding cost per customer per unit time

C_l = Cost associated to each lost customer per unit time

C_r = Cost associated to each reneged customer per unit time

C_R = Cost associated to each retained customer per unit time

C_R = Earned revenue by providing service to each customer

TEC = Total expected cost of the system
TER = Total expected revenue of the system
TEP = Total expected profit of the system

For a finite capacity system some customers cannot join the system when they find that the system is full, then immediately they go elsewhere and said to be lost from the system with rate $\lambda_{lost} = \lambda P_N$, where

$$P_N = \frac{1}{(N-c+1)!}\prod_{k=c+1}^{N}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{\lambda^c}{c!\mu^c}P_0$$

We can obtain the average reneging rate R_r and the average retention rate R_R as follows:

$$R_r = \sum_{n=1}^{N}\left(n-c\right)\xi p P_n \qquad \text{and} \qquad R_R = \sum_{n=1}^{N}\left(n-c\right)\xi q P_n.$$

We define the total expected cost (TEC) of the system as:

$$TEC = C_s\mu + C_h L_s + C_l\lambda P_N + C_r C_r + C_R C_R$$

$$TEC = C_s\mu + C_h\left[\sum_{n=1}^{c} n\frac{\lambda^n}{n!\mu^n}P_0 + \sum_{n=c+1}^{N} n\frac{1}{(n-c+1)!}\prod_{k=c+1}^{n}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{\lambda^c}{c!\mu^c}P_0\right]$$
$$+C_l\lambda\frac{1}{(N-c+1)!}\prod_{k=c+1}^{N}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{\lambda^c}{c!\mu^c}P_0$$
$$+C_R\sum_{n=c+1}^{N}\left(n-c\right)\xi q\frac{1}{(N-c+1)!}\prod_{k=c+1}^{n}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{\lambda^c}{c!\mu^c}P_0$$

Let R be the earned revenue by providing service to each customer per unit time, then RL_s would be total earned revenue for providing service to average number of customers in the system; also, $R\lambda P_N$ and RR_r would be the loss in the revenue of the system due to number of lost customers per unit time and due to reneging of customers per unit time respectively. Hence, total expected revenue (TER) of the system is given by:

$$TER = RL_S - R\lambda P_N - RR_r$$

$$TER = R\left[\sum_{n=1}^{c} n\frac{\lambda^n}{n!\mu^n}P_0 + \sum_{n=c+1}^{N} n\frac{1}{(n-c+1)!}\prod_{k=c+1}^{n}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{\lambda^c}{c!\mu^c}P_0\right]$$
$$-R\lambda\frac{1}{(N-c+1)!}\prod_{k=c+1}^{N}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{\lambda^c}{c!\mu^c}P_0$$
$$-R\sum_{n=c+1}^{N}\left(n-c\right)\xi p\frac{1}{(N-c+1)!}\prod_{k=c+1}^{n}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{\lambda^c}{c!\mu^c}P_0$$

Now, total expected profit (TEP) of the system is defined as:

$$TEP = TER - TEC$$

Thus,

$$TEP = \left(R - C_h\right)L_s - (R + C_l)\lambda P_N - (R + C_r)R_r - \mu C_s - C_R R_R$$

$$TEP = \left(R - C_h\right)\left[\sum_{n=1}^{c} n\frac{\lambda^n}{n!\mu^n}P_0 + \sum_{n=c+1}^{N} n\frac{1}{(n-c+1)!}\prod_{k=c+1}^{n}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{\lambda^c}{c!\mu^c}P_0\right]$$
$$-(R + C_h)\left[\sum_{n=1}^{c} n\frac{\lambda^n}{n!\mu^n}P_0 + \sum_{n=c+1}^{N} n\frac{1}{(n-c+1)!}\prod_{k=c+1}^{n}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{\lambda^c}{c!\mu^c}P_0\right]$$
$$-(R + C_r)\sum_{n=c+1}^{N}\left(n-c\right)\xi p\frac{1}{(N-c+1)!}\prod_{k=c+1}^{n}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{\lambda^c}{c!\mu^c}P_0 - \mu C_s$$
$$-C_R\sum_{n=c+1}^{N}\left(n-c\right)\xi q\frac{1}{(N-c+1)!}\prod_{k=c+1}^{n}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{\lambda^c}{c!\mu^c}P_0$$

6.1. Cost-profit Analysis of M/M/c/N queue with customer retention and discouragement with respect to effect of effect of probability of customer retention on total expected revenue, total expected cost and total expected profit of the model

In this sub-section, the impact of probability of customer retention on total expected cost and profit of the model is studied. From table-2, we observe that total expected cost as well as total expected profit of the system increase as the probability of customer retention increases. The retention of reneged customers contributes to the total expected revenue of the system and it increases as the probability of customer retention increases. The increase in total expected cost with the increase in probability of customer retention may be attributed to the increasing holding and customer retention costs.

Table 2. Impact of probability of customer retention (q) on the total expected profit (Here, λ=2, μ=3, ξ = 0.1, N=4, c=2, C_s=20, C_h=10, C_l =25, C_r=8, C_R=10, and R=300)

Probability of retaining the impatient customers (q)	Total Expected Revenue (TER)	Total Expected Cost (TEC)	Total Expected Profit (TEP)
0	162.7678	65.77018	96.99763
0.05	162.9423	65.77214	97.17017
0.1	163.117	65.7741	97.34287
0.15	163.2918	65.77606	97.51574
0.2	163.4668	65.77803	97.68877
0.25	163.642	65.78	97.86196
0.3	163.8173	65.78197	98.03531
0.35	163.9928	65.78395	98.20882
0.4	164.1684	65.78593	98.3825
0.45	164.3443	65.78791	98.55635
0.5	164.5203	65.7899	98.73035
0.55	164.6964	65.79189	98.90453
0.6	164.8727	65.79388	99.07887
0.65	165.0493	65.79588	99.25338
0.7	165.2259	65.79788	99.42805
0.75	165.4028	65.79988	99.60289
0.8	165.5798	65.80189	99.7779
0.85	165.757	65.8039	99.95308
0.9	165.9343	65.80591	100.1284
0.95	166.1119	65.80793	100.3039
1	166.2896	65.80995	100.4796

6.2. Cost-profit Analysis of M/M/c/N queue with customer retention, and M/M/c/N queue with customer retention and discouragement (a comparison)

In this sub-section, the relative comparison of two queueing models M/M/c/N queue with customer retention and M/M/c/N queue with customer retention and discouragement is performed by studying the variation in TEC, TER and TEP of these models with the change in probability of customer retention. Such comparisons are shown in figure 2, 3 and 4. From these figures, we find that the TEC, TER and TEP of M/M/c/N queue with customer retention remain always higher than that of the M/M/c/N queue with customer retention and discouragement. Thus, a decision maker can study the impact of customer discouragement on the cost and profit aspects of the queueing system.

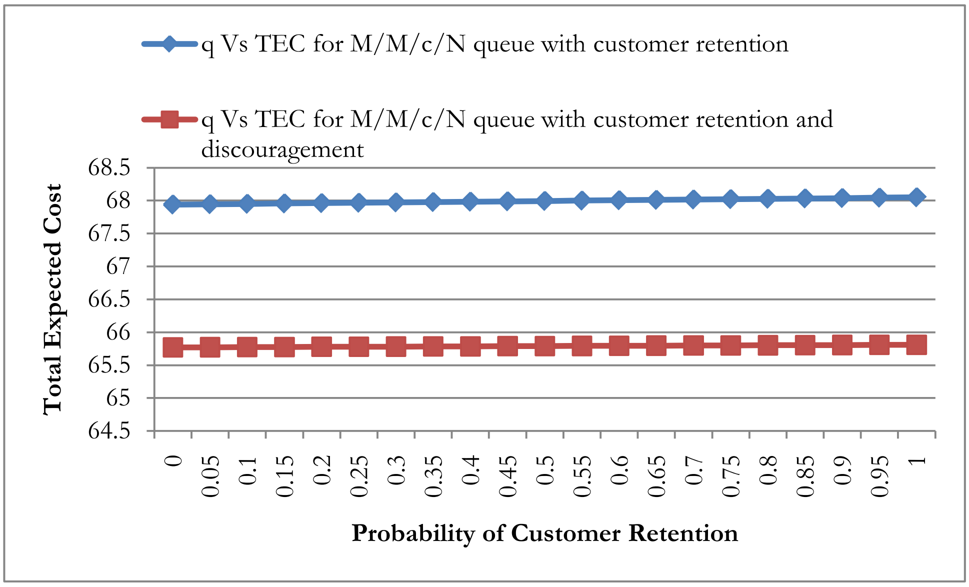

Figure 2. Change in total expected cost with respect to probability of customer retention

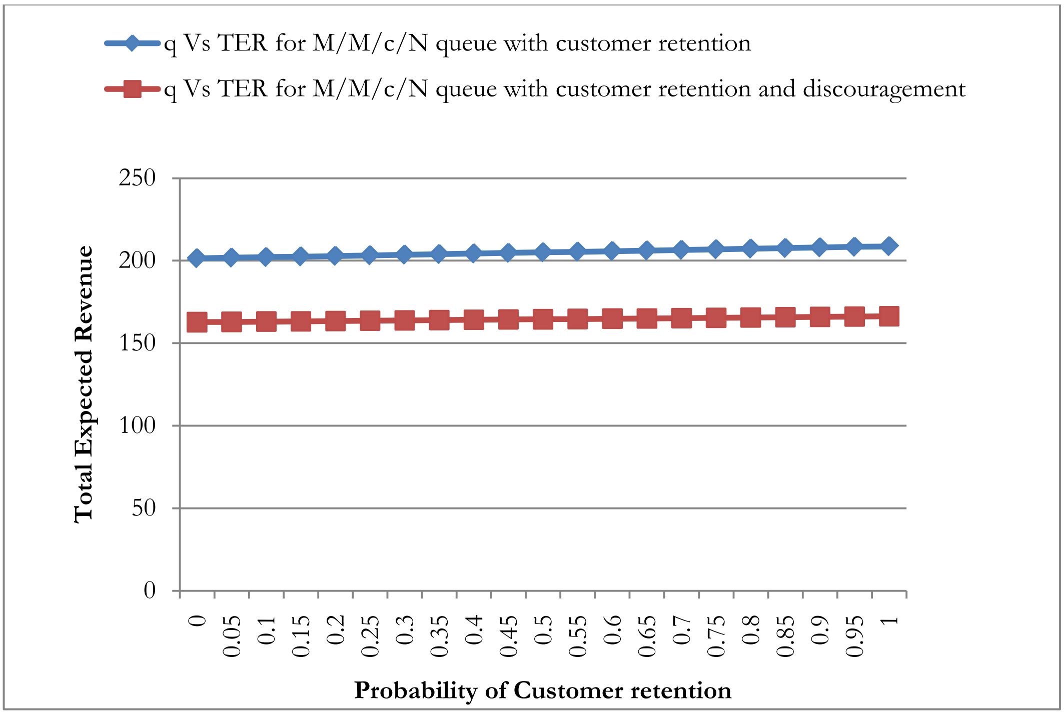

Figure 3. Change in total expected revenue with respect to probability of customer retention

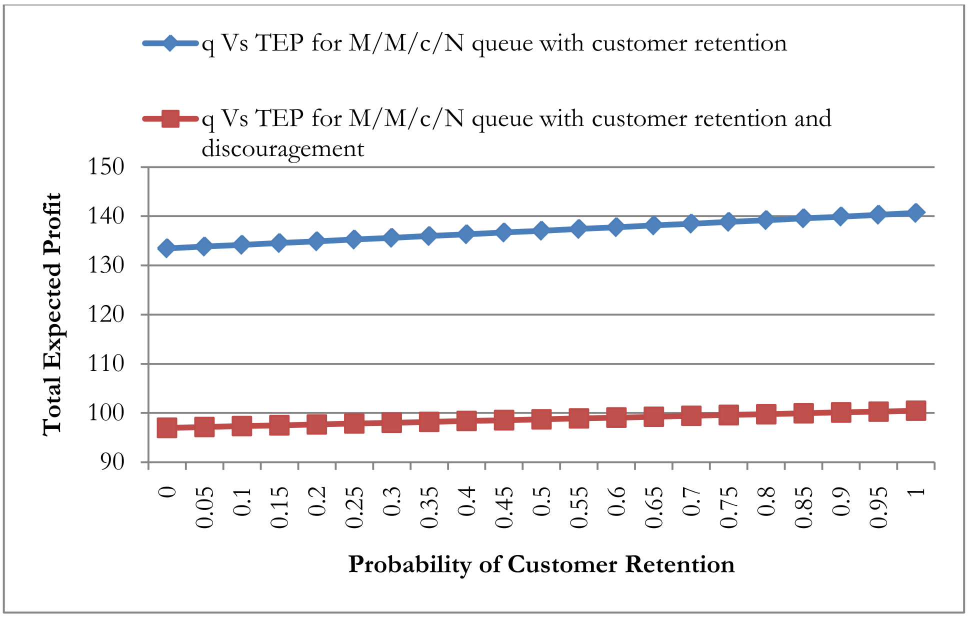

Figure 4. Change in total expected profit with respect to probability of customer retention

7. SPECIAL CASES

In this section, we derive and discuss some important special cases of the model under consideration.

1. When there is no customer discouragement

The model reduces to an M / M /c / N queueing system with reneging and retention of reneged customers as studied by Kumar and Sharma (2013) with

$$P_n = \begin{cases} \dfrac{\lambda^n}{n!\mu^n} P_0; 1 \le n \le c \\ \prod_{k=c+1}^{n} \dfrac{\lambda}{c\mu + (k-c)\xi p} \dfrac{\lambda^c}{c!\mu^c} P_0; c+1 \le n \le N \end{cases}$$

and P_0 is given by

$$P_0 = \frac{1}{\left(1 + \sum_{n=1}^{c} \dfrac{\lambda^n}{n!\mu^n} + \sum_{n=c+1}^{N} \prod_{k=c+1}^{n} \dfrac{\lambda}{c\mu + (k-c)\xi p} \dfrac{\lambda^c}{c!\mu^c}\right)}.$$

2. When there is no customer retention (i.e. q=0).

In this case, the queueing model gets reduces to an M/M/c/N queueing model with discouraged arrivals and reneging with

$$P_n = \begin{cases} \dfrac{\lambda^n}{n!\mu^n} P_0; 1 \le n \le c \\ \dfrac{1}{(n-c+1)!} \prod_{k=c+1}^{n} \dfrac{\lambda}{c\mu + (k-c)\xi} \dfrac{\lambda^c}{c!\mu^c} P_0; c+1 \le n \le N \end{cases}$$

and P_0 is given by

$$P_0 = \frac{1}{\left(1 + \sum_{n=1}^{c} \dfrac{\lambda^n}{n!\mu^n} + \sum_{n=c+1}^{N} \dfrac{1}{(n-c+1)!} \prod_{k=c+1}^{n} \dfrac{\lambda}{c\mu + (k-c)\xi} \dfrac{\lambda^c}{c!\mu^c}\right)}$$

3. When there is only one server and no customer discouragement:

In this case, the model resembles with the one studied by Kumar and Sharma (2012d) with

$$P_n = \prod_{k=1}^{n} \frac{\lambda}{\mu + (k-1)\xi p} P_0; 1 \le n \le N$$

and $$P_0 = \frac{1}{\left(1 + \sum_{n=1}^{N} \prod_{k=1}^{n} \frac{\lambda}{\mu + (k-1)\xi p}\right)}$$

4. When there is only one server and no customer retention

In this case, the model resembles with the one studied by Ammar *et al.* (2012).

8. CONCLUSIONS AND FUTURE WORK

This paper studies an M/M/c/N queueing model with discouraged arrivals, reneging and retention of reneged customers. The stationary probabilities of the system size are derived explicitly. The cost-profit model is presented and some useful comparisons are also performed. Finally, some significant queueing models are derived as special cases of this model. The queueing model studied in this paper finds its applications in computer communications, manufacturing, and in hospital management.

The model analysis is limited to finite capacity. The infinite capacity case of the model can also be studied. Further, the model can be solved in transient state to get time-dependent results. The same idea can be extended to some non-Markovian queueing models.

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